Why Functions as Values

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  - abstract over **`add`** and **`sub`** cases
  - **`filter`**, **`map`**, etc.

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Why Functions as Values

- Abstraction is easier with functions as values
  - abstract over **`add`** and **`sub`** cases
  - **`filter`**, **`map`**, etc.
- What are objects? Callbacks?
- Separate **`deffun`** form becomes unnecessary
  - ```
    {deffun {f x} {+ 1 x}}
    {f 10}
    ⇒
    {with {f {fun {x} {+ 1 x}}}
      {f 10}}
    ```
    ```

FWAE Grammar, Almost

```
<FWAE>  ::=  <num>
          |  {+ <FWAE> <FWAE>}
          |  {- <FWAE> <FWAE>}
          |  {with {<id> <FWAE>} <FWAE>}
          |  <id>
          |  {<id> <FWAE>}                    ?
          |  {fun {<id>} <FWAE>}              NEW
```

FWAE Evaluation

```
10 ⇒ 10

{+ 1 2} ⇒ 3

{- 1 2} ⇒ -1

{with {x 7} {+ x 2}} ⇒ {+ 7 2} ⇒ 9

y ⇒ free variable
```

FWAE Evaluation

```
10 ⇒ 10

{+ 1 2} ⇒ 3

{- 1 2} ⇒ -1

{with {x 7} {+ x 2}} ⇒ {+ 7 2} ⇒ 9

y ⇒ free variable

{fun {x} {+ 1 x}} ⇒
```

FWAE Evaluation

```
10 ⇒ 10

{+ 1 2} ⇒ 3

{- 1 2} ⇒ -1

{with {x 7} {+ x 2}} ⇒ {+ 7 2} ⇒ 9

y ⇒ free variable

{fun {x} {+ 1 x}} ⇒ {fun {x} {+ 1 x}}
```

FWAE Evaluation

```
10 ⇒ 10

{+ 1 2} ⇒ 3

{- 1 2} ⇒ -1

{with {x 7} {+ x 2}} ⇒ {+ 7 2} ⇒ 9

y ⇒ free variable

{fun {x} {+ 1 x}} ⇒ {fun {x} {+ 1 x}}
```

Result is not always a number!

```
; interp FWAE ... -> FWAE-Value
```

FWAE Evaluation

```
{with {y 10} {fun {x} {+ y x}}}
```

$\Rightarrow$

FWAE Evaluation

```
{with {y 10} {fun {x} {+ y x}}}
```

$\Rightarrow$ `{fun {x} {+ 10 x}}`

FWAE Evaluation

```
    {with {y 10} {fun {x} {+ y x}}}
⇒  {fun {x} {+ 10 x}}

{with {f {fun {x} {+ 1 x}}}
  {f 3}}
⇒
```

FWAE Evaluation

```
    {with {y 10} {fun {x} {+ y x}}}
```

```
 ⇒  {fun {x} {+ 10 x}}
```

```
{with {f {fun {x} {+ 1 x}}}
  {f 3}}
```

```
 ⇒  {{fun {x} {+ 1 x}} 3}
```

FWAE Evaluation

```
    {with {y 10} {fun {x} {+ y x}}}
```

⇒ `{fun {x} {+ 10 x}}`

```
{with {f {fun {x} {+ 1 x}}}
  {f 3}}
```

⇒ `{{fun {x} {+ 1 x}} 3}`

Doesn't match the grammar for **<FWAE>**

FWAE Grammar

```
<FWAE>  ::=  <num>
         |   {+ <FWAE> <FWAE>}
         |   {- <FWAE> <FWAE>}
         |   {with {<id> <FWAE>} <FWAE>}
         |   <id>
         |   ~~{<id> <FWAE>}~~
         |   {fun {<id>} <FWAE>}        NEW
         |   {<FWAE> <FWAE>}            NEW
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4
```

```
{{fun {x} {+ 1 x}} 3}  ⇒  {+ 1 3}  ⇒  4
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4

{{fun {x} {+ 1 x}} 3}  ⇒  {+ 1 3}  ⇒  4

{1 2}  ⇒
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4

{{fun {x} {+ 1 x}} 3}  ⇒  {+ 1 3}  ⇒  4

{1 2}  ⇒  not a function
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4

{{fun {x} {+ 1 x}} 3}  ⇒  {+ 1 3}  ⇒  4

{1 2}  ⇒  not a function

{+ 1 {fun {x} 10}}  ⇒
```

FWAE Evaluation

```
{with {f {fun {x} {+ 1 x}}} {f 3}}
 ⇒  {{fun {x} {+ 1 x}} 3}
 ⇒  {+ 1 3}  ⇒  4

{{fun {x} {+ 1 x}} 3}  ⇒  {+ 1 3}  ⇒  4

{1 2}  ⇒  not a function

{+ 1 {fun {x} 10}}  ⇒  not a number
```

FWAE Datatype

```
(define-type FWAE
  [num (n number?)]
  [add (lhs FWAE?)
       (rhs FWAE?)]
  [sub (lhs FWAE?)
       (rhs FWAE?)]
  [with (name symbol?)
        (named-expr FWAE?)
        (body FWAE?)]
  [id (name symbol?)]
  [fun (param symbol?)
       (body FWAE?)]
  [app (fun-expr FWAE?)
       (arg-expr FWAE?)])
```

FWAE Datatype

```
(define-type FWAE
  [num (n number?)]
  [add (lhs FWAE?)
       (rhs FWAE?)]
  [sub (lhs FWAE?)
       (rhs FWAE?)]
  [with (name symbol?)
        (named-expr FWAE?)
        (body FWAE?)]
  [id (name symbol?)]
  [fun (param symbol?)
       (body FWAE?)]
  [app (fun-expr FWAE?)
       (arg-expr FWAE?)])
```

```
(test (parse '{fun {x} {+ x 1}})
      (fun 'x (add (id 'x) (num 1))))
```

FWAE Datatype

```
(define-type FWAE
  [num (n number?)]
  [add (lhs FWAE?)
       (rhs FWAE?)]
  [sub (lhs FWAE?)
       (rhs FWAE?)]
  [with (name symbol?)
        (named-expr FWAE?)
        (body FWAE?)]
  [id (name symbol?)]
  [fun (param symbol?)
       (body FWAE?)]
  [app (fun-expr FWAE?)
       (arg-expr FWAE?)])
```

```
(test (parse '{{fun {x} {+ x 1}} 10})
      (app (fun 'x (add (id 'x) (num 1))) (num 10)))
```

FWAE Interpreter

```
; interp : FWAE -> FWAE
(define (interp a-wae)
  (type-case FWAE a-wae
    [num (n) a-wae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr)
         ...]
    [app (fun-expr arg-expr)
         ...]))
```

FWAE Interpreter

```
; interp : FWAE -> FWAE
(define (interp a-wae)
  (type-case FWAE a-wae
    [num (n) a-wae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr)
         a-wae]
    [app (fun-expr arg-expr)
         ...]))
```

FWAE Interpreter

```
; interp : FWAE -> FWAE
(define (interp a-wae)
  (type-case FWAE a-wae
    [num (n) a-wae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr)
         a-wae]
    [app (fun-expr arg-expr)
         ... (interp fun-expr)
         ... (interp arg-expr) ...]))
```

FWAE Interpreter

```
; interp : FWAE -> FWAE
(define (interp a-wae)
  (type-case FWAE a-wae
    [num (n) a-wae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr)
         a-wae]
    [app (fun-expr arg-expr)
         (local [(define fun-val (interp fun-expr))]
           ... (fun-body fun-val) ...
           ... (fun-param fun-val) ...
           ... (interp arg-expr) ...)]))
```

FWAE Interpreter

```
; interp : FWAE -> FWAE
(define (interp a-wae)
  (type-case FWAE a-wae
    [num (n) a-wae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr)
         a-wae]
    [app (fun-expr arg-expr)
         (local [(define fun-val (interp fun-expr))]
           (interp (subst (fun-body fun-val)
                          (fun-param fun-val)
                          (interp arg-expr))))]))
```

Add and Subtract

```
; num+ : FWAE-Value FWAE-Value -> FWAE-Value
(define (num+ x y)
  (numV (+ (numV-n x) (numV-n y))))

; num- : FWAE-Value FWAE-Value -> FWAE-Value
(define (num- x y)
  (numV (- (numV-n x) (numV-n y))))
```

Add and Subtract

```
; num+ : FWAE-Value FWAE-Value -> FWAE-Value
(define (num+ x y)
  (numV (+ (numV-n x) (numV-n y))))

; num- : FWAE-Value FWAE-Value -> FWAE-Value
(define (num- x y)
  (numV (- (numV-n x) (numV-n y))))
```

Better:

```
; num-op : (num num -> num) -> (FWAE-Value FWAE-Value -> FWA
(define (num-op op)
  (lambda (x y)
    (numV (op (numV-n x) (numV-n y)))))

(define num+ (num-op +))
(define num- (num-op -))
```

FWAE Subst

```
; subst : FWAE symbol FWAE -> FWAE
(define (subst exp sub-id val)
  (type-case FWAE exp
    ...
    [id (name)
      (cond
        [(equal? name sub-id) val]
        [else exp])]
    [app (f arg)
         (app (subst f sub-id val)
              (subst arg sub-id val))]
    [fun (id body)
         (if (equal? sub-id id)
             exp
             (fun id (subst body sub-id val)))]))
```

FWAE Subst

Beware: with the implementation on the previous slide,

```
(subst {with {y 10} z}
       'z
       {fun {x} {+ x y}})
```

⇒ `{with {y 10} {fun {x} {+ x y}}}`

which is wrong, but we ignore this problem

- Only happens when the original program has free variables
- The problem disappears with deferred substitution, anyway

No More With

Compare the **with** and **app** implementations:

```
(define (interp a-wae)
  (type-case FWAE a-wae
    ...
    [with (bound-id named-expr body-expr)
      (interp (subst body-expr
                     bound-id
                     (interp named-expr)))]
    ...
    [app (fun-expr arg-expr)
         (local [(define fun-val (interp fun-expr))]
           (interp (subst (fun-body fun-val)
                          (fun-param fun-val)
                          (interp arg-expr))))]))
```

The **app** case does everything that **with** does

No More With

```
{with {x 10} x}
```

is the same as

```
{{fun {x} x} 10}
```

No More With

```
{with {x 10} x}
```

is the same as

```
{{fun {x} x} 10}
```

In general,

```
{with {<id> <FWAE>1} <FWAE>2}
```

is the same as

```
{{fun {<id>} <FWAE>2} <FWAE>1}
```

No More With

```
{with {x 10} x}
```

is the same as

```
{{fun {x} x} 10}
```

In general,

```
{with {<id> <FWAE>1} <FWAE>2}
```

is the same as

```
{{fun {<id>} <FWAE>2} <FWAE>1}
```

Let's assume

```
(test {with {<id> <FWAE>1} <FWAE>2}
      (app (fun '<id> <FWAE>2) <FWAE>1))
```

FAE Grammar

```
<FAE>  ::=  <num>
         |  {+ <FAE> <FAE>}
         |  {- <FAE> <FAE>}
         |  {with {<id> <FAE>} <FAE>}
         |  <id>
         |  {fun {<id>} <FAE>}
         |  {<FAE> <FAE>}
```

FAE Grammar

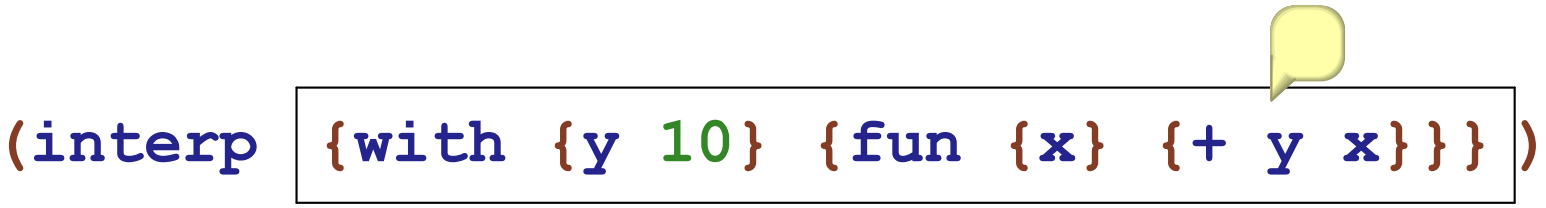
```
<FAE>  ::=  <num>
         |  {+ <FAE> <FAE>}
         |  {- <FAE> <FAE>}
         |  {with {<id> <FAE>} <FAE>}
         |  <id>
         |  {fun {<id>} <FAE>}
         |  {<FAE> <FAE>}
```

- We'll still use **`with`** in boxes
- No more case lines in **`interp`**, etc. for **`with`**
- No more test cases for **`interp`**, etc. using **`with`**

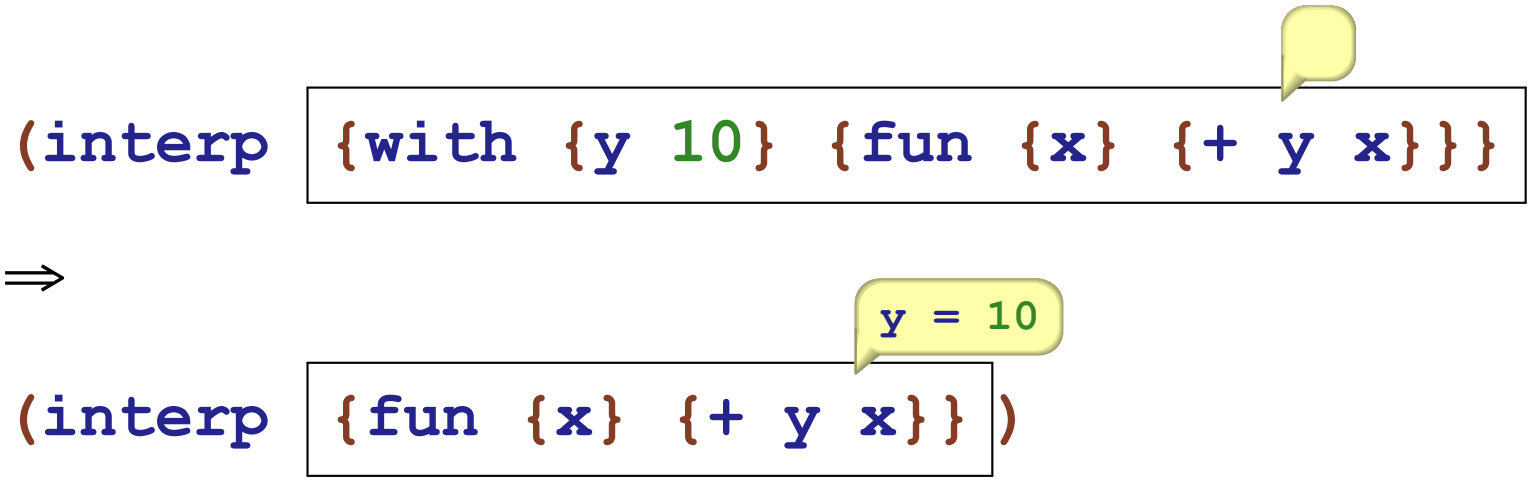
FAE Interpreter

```
; interp : FAE -> FAE
(define (interp a-fae)
  (type-case FAE a-wae
    [num (n) a-fae]
    [add (l r) (num+ (interp l) (interp r))]
    [sub (l r) (num- (interp l) (interp r))]
    [id (name) (error 'interp "free variable")]
    [fun (param body-expr) a-fae]
    [app (fun-expr arg-expr)
         (local [(define fun-val (interp fun-expr))]
           (interp (subst (fun-body fun-val)
                          (fun-param fun-val)
                          (interp arg-expr))))]))
```

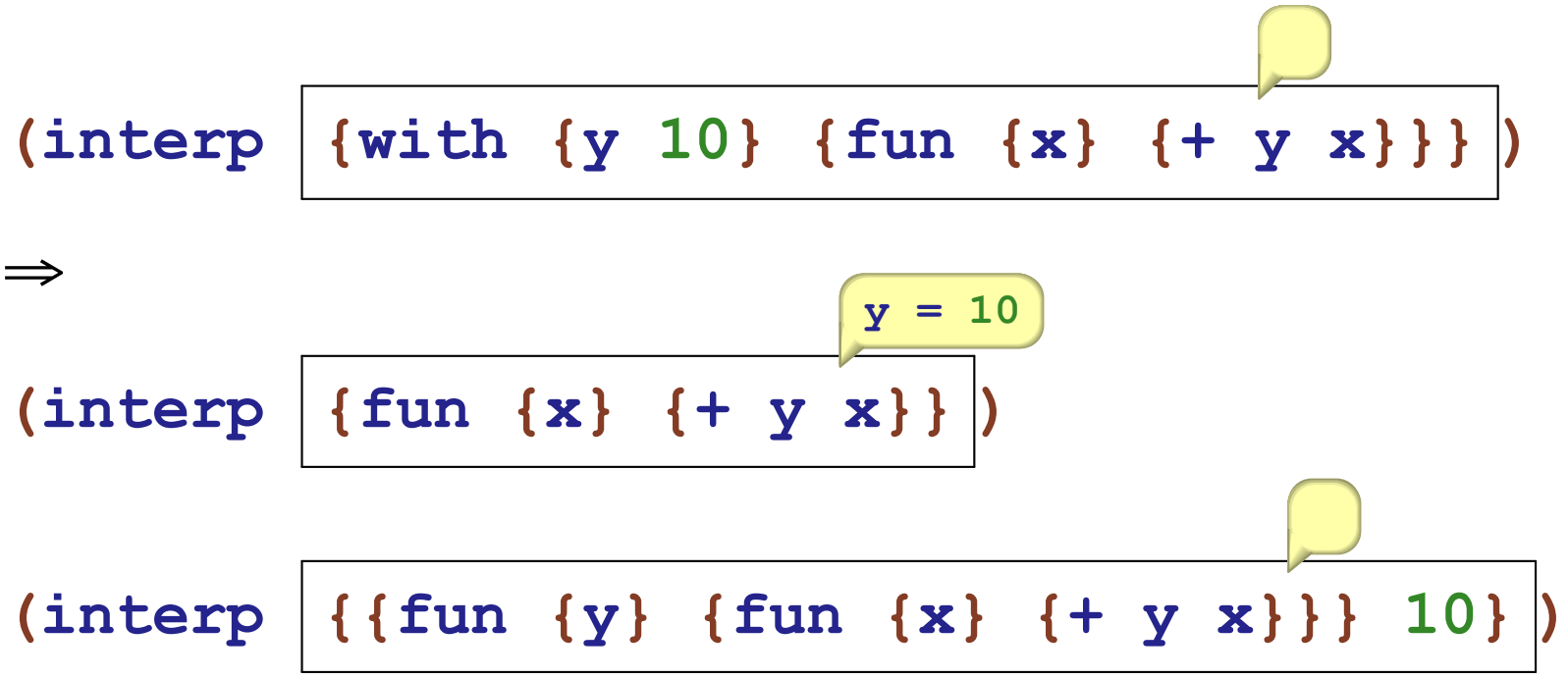
FAE with Deferred Substitution



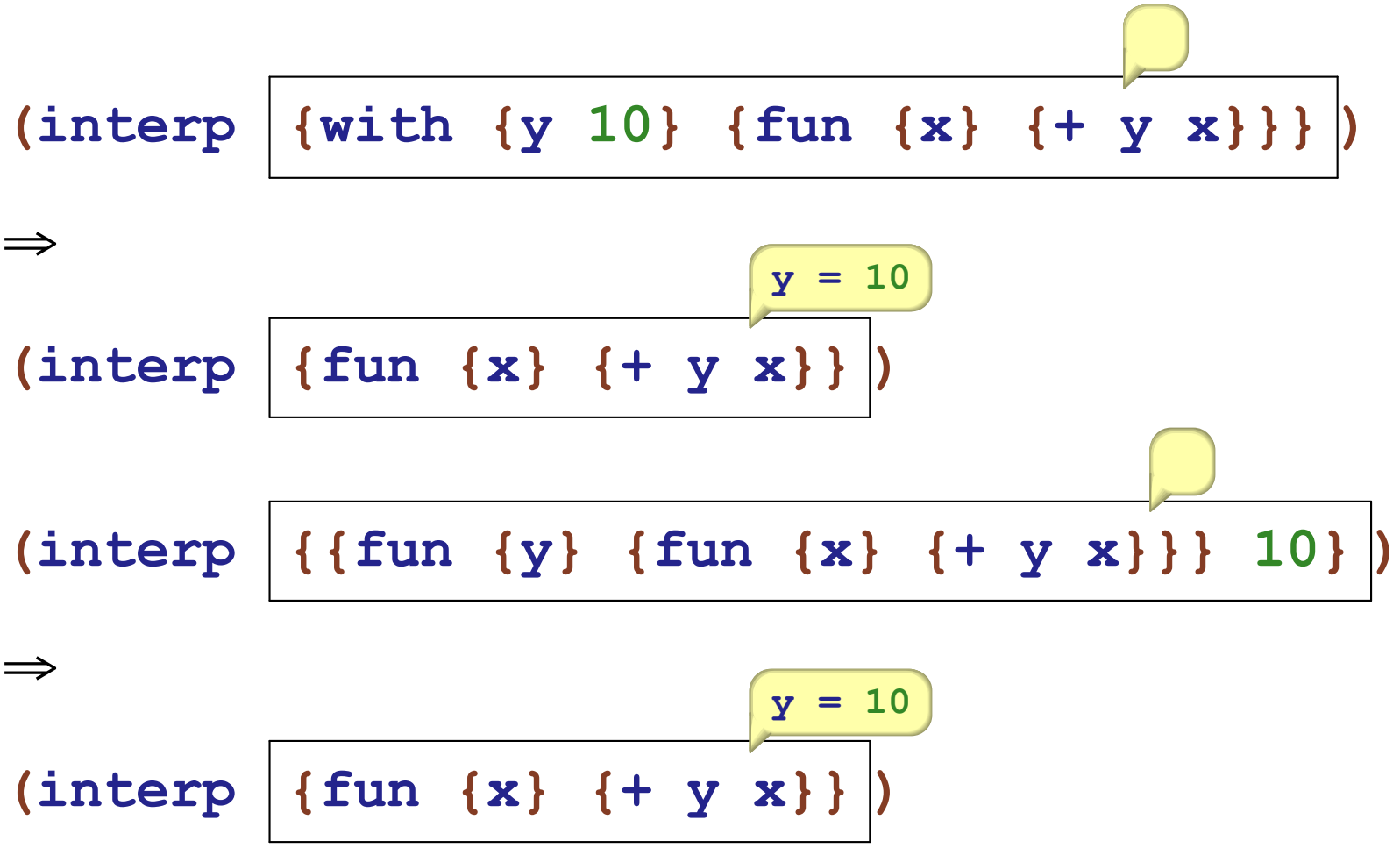
FAE with Deferred Substitution



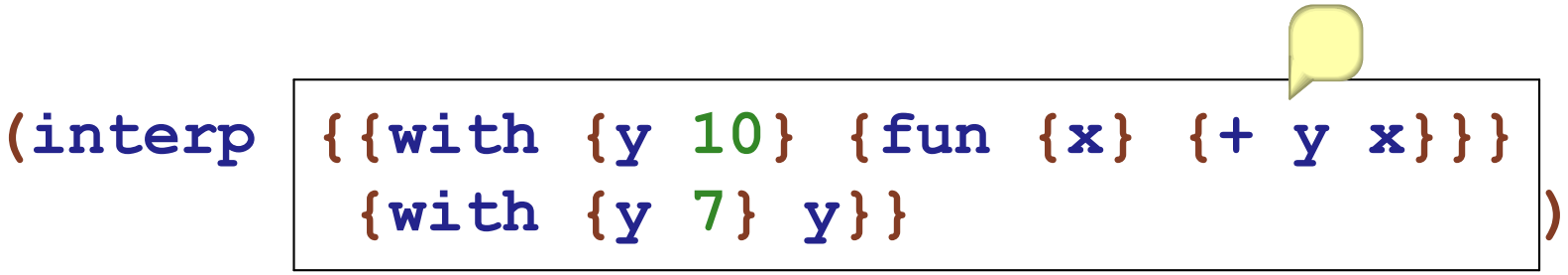
FAE with Deferred Substitution



FAE with Deferred Substitution



FAE with Deferred Substitution



FAE with Deferred Substitution

```
(interp {{with {y 10} {fun {x} {+ y x}}}
         {with {y 7} y}})
```

Argument expression:

```
(interp {with {y 7} y})
```

⇒

y = 7

```
(interp y) ⇒ 7
```

FAE with Deferred Substitution

```
(interp {{with {y 10} {fun {x} {+ y x}}}
         {with {y 7} y}})
```

Argument expression:

```
(interp {with {y 7} y})
```

⇒

y = 7

```
(interp y) ⇒ 7
```

Function expression:

```
(interp {with {y 10} {fun {x} {+ y x}}})
```

⇒

y = 10

```
(interp {fun {x} {+ y x}}) ⇒ ?
```

FAE Values

A function value needs to keep its substitution cache

```
(define-type FWAE-Value
  [numV (n number?)]
  [closureV (param symbol?)
            (body FAE?)
            (ds DefrdSub?)])

(define-type DefrdSub
  [mtSub]
  [aSub (name symbol?)
        (value FWAE-Value?)
        (ds DefrdSub?)])
```

FAE Values

A function value needs to keep its substitution cache

```
(define-type FWAE-Value
  [numV (n number?)]
  [closureV (param symbol?)
            (body FAE?)
            (ds DefrdSub?)])

(define-type DefrdSub
  [mtSub]
  [aSub (name symbol?)
        (value FWAE-Value?)
        (ds DefrdSub?)])
```

```
(test (interp {with {y 10} {fun {x} {+ y x}}})
      ...)
```

FAE Values

A function value needs to keep its substitution cache

```
(define-type FWAE-Value
  [numV (n number?)]
  [closureV (param symbol?)
            (body FAE?)
            (ds DefrdSub?)])

(define-type DefrdSub
  [mtSub]
  [aSub (name symbol?)
        (value FWAE-Value?)
        (ds DefrdSub?)])
```

```
(test (interp {with {y 10} {fun {x} {+ y x}}})
      (closureV ... ... ...))
```

FAE Values

A function value needs to keep its substitution cache

```
(define-type FWAE-Value
  [numV (n number?)]
  [closureV (param symbol?)
            (body FAE?)
            (ds DefrdSub?)])

(define-type DefrdSub
  [mtSub]
  [aSub (name symbol?)
        (value FWAE-Value?)
        (ds DefrdSub?)])
```

```
(test (interp {with {y 10} {fun {x} {+ y x}}})
      (closureV 'x {+ y x}
                (aSub 'y (num 10) (mtSub))))
```

Continuing Evaluation

Function: `{fun {x} {+ y x}}` with `y = 10`

Argument: 7

Continuing Evaluation

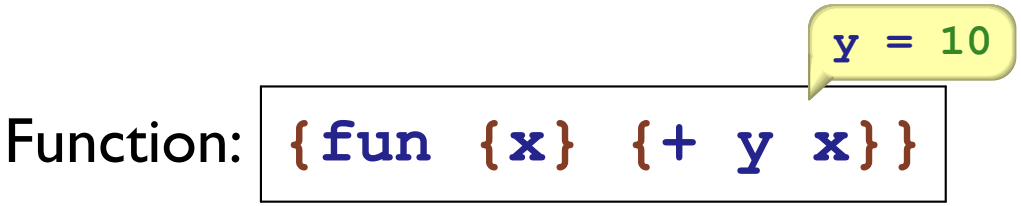
Function: `{fun {x} {+ y x}}` with `y = 10`

Argument: `7`

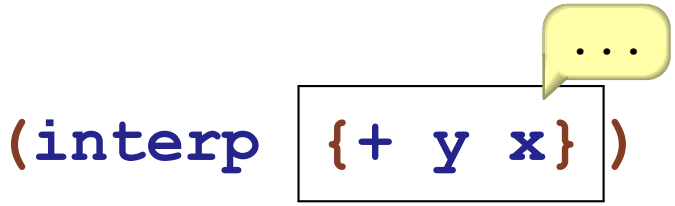
To apply, interpret the function body with the given argument:

```
(interp ...)
```

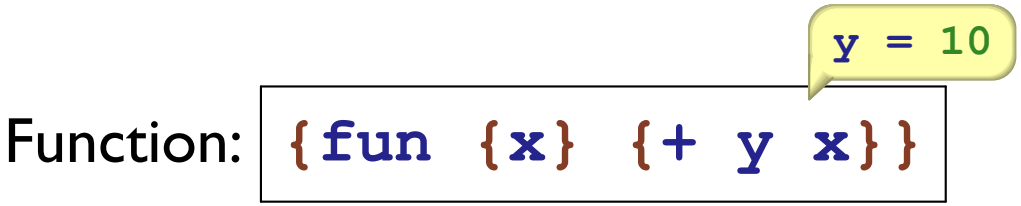
Continuing Evaluation



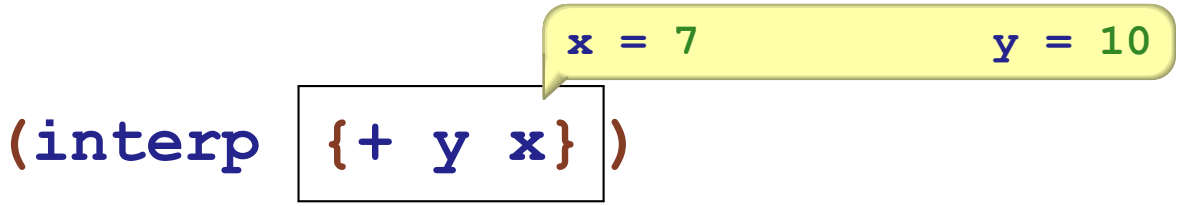
To apply, interpret the function body with the given argument:



Continuing Evaluation



To apply, interpret the function body with the given argument:



FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         ...]
    [app (fun-expr arg-expr)
         ...]))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         ...]))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         ... (interp fun-expr ds)
         ... (interp arg-expr ds) ...]))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         (local [(define fun-val
                   (interp fun-expr ds))
                 (define arg-val
                   (interp arg-expr ds))]
           ...)]))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         (local [(define fun-val
                   (interp fun-expr ds))
                 (define arg-val
                   (interp arg-expr ds))]
           (interp (closureV-body fun-val)
                   ...))])))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         (local [(define fun-val
                   (interp fun-expr ds))
                 (define arg-val
                   (interp arg-expr ds))]
           (interp (closureV-body fun-val)
                   (aSub (closureV-param fun-val)
                         arg-val
                         ...)))])))
```

FAE Interpreter with Substitution

```
; interp : FAE DefrdSub -> FWAE-Value
(define (interp a-wae ds)
  (type-case FAE a-wae
    [num (n) (numV n)]
    [add (l r) (num+ (interp l ds) (interp r ds))]
    [sub (l r) (num- (interp l ds) (interp r ds))]
    [id (name) (lookup name ds)]
    [fun (param body-expr)
         (closureV param body-expr ds)]
    [app (fun-expr arg-expr)
         (local [(define fun-val
                   (interp fun-expr ds))
                 (define arg-val
                   (interp arg-expr ds))]
           (interp (closureV-body fun-val)
                   (aSub (closureV-param fun-val)
                         arg-val
                         (closureV-ds fun-val))))])))
```