What are type rules?

$$\frac{\Gamma \vdash \mathbf{e}_1 : \mathit{num} \qquad \Gamma \vdash \mathbf{e}_2 : \mathit{num}}{\Gamma \vdash \{+\ \mathbf{e}_1\ \mathbf{e}_2\} : \mathit{num}}$$

An example - the rule for **+**

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An example - the rule for **+**

- This is just one of a set of *inference rules*.
- Together the set of rules define the *type judgment*, which is a relation that assigns types to expressions.

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An example - the rule for **+**

- This is just one of a set of *inference rules*.
- Together the set of rules define the *type judgment*, which is a relation that assigns types to expressions.
- Fine, but what does that mean...

Inference rules

$$\frac{\mathbf{A} \qquad\qquad \mathbf{B}}{\mathbf{C}}$$

The general form of a inference rule

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The general form of a inference rule

- A and B are *premises* (not necessarily two of them)
- C is the *conclusion*

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The general form of a inference rule

- A and B are *premises* (not necessarily two of them)
- C is the *conclusion*
- This is a *rule*, which says:
- If I know A and B, then I can conclude C

Inference rules

$$\frac{\mathbf{A} \qquad\qquad \mathbf{B}}{\mathbf{A} \land \mathbf{B}}$$

An example — logical "and"

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An example — logical "and"

- If I know **A**, and I know **B**, then I can conclude **A ∧ B**

Inference rules

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An example — logical "and"

- If I know **A**, and I know **B**, then I can conclude **A ∧ B**
- How would I know **A** and **B**?

Inference rules

$$\frac{\mathbf{A} \qquad \mathbf{B}}{\mathbf{A} \land \mathbf{B}}$$

An example — logical "and"

- If I know **A**, and I know **B**, then I can conclude $\mathbf{A} \land \mathbf{B}$
- How would I know **A** and **B**?
- I used some rule to conclude they were true

Inference rules

$$\frac{\mathbf{A} \qquad \mathbf{B}}{\mathbf{A} \wedge \mathbf{B}}$$

An example — logical "and"

- If I know $\mathbf{A}$, and I know $\mathbf{B}$, then I can conclude $\mathbf{A} \wedge \mathbf{B}$
- Some other $\wedge$ rules:

$$\frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{A}} \qquad \frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{B}}$$

Inference rules

$$\frac{\mathbf{A} \qquad \mathbf{B}}{\mathbf{A} \wedge \mathbf{B}}$$

An example — logical "and"

- If I know $\mathbf{A}$, and I know $\mathbf{B}$, then I can conclude $\mathbf{A} \wedge \mathbf{B}$

$$\frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{A}} \qquad \frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{B}}$$

- Add some more rules and you have a system for deciding the truth of logical sentences

A TFAE Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : \mathit{num} \qquad \Gamma \vdash \mathbf{e}_2 : \mathit{num}}{\Gamma \vdash \{+\ \mathbf{e}_1\ \mathbf{e}_2\} : \mathit{num}}$$

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One of a set of rules that define the *type judgment*

$$\Gamma \vdash \mathbf{e} : \tau$$

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- Which means...
- With the type bindings in Γ, I can conclude that $\mathbf{e}$ has the type τ

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One of a set of rules that define the *type judgment*

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- Which means...
- With the type bindings in Γ, I can conclude that **e** has the type τ
- Γ is the *type environment* and is just a map from **<id>** to τ (type)

TFAE Rules

$$\Gamma \vdash \texttt{<num>} : \textit{num} \qquad [\ldots\ \texttt{<id>}\leftarrow\tau\ \ldots] \vdash \texttt{<id>} : \tau$$

$$\Gamma \vdash \texttt{true} : \textit{bool} \qquad \Gamma \vdash \texttt{false} : \textit{bool}$$

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$$\frac{\Gamma[\texttt{<id>}\leftarrow\tau_1] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{\texttt{fun}\ \{\texttt{<id>} : \tau_1\}\ \mathbf{e}\} : (\tau_1 \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \rightarrow \tau_0) \qquad \Gamma \vdash \mathbf{e}_1 : \tau_1}{\Gamma \vdash \{\mathbf{e}_0\ \mathbf{e}_1\} : \tau_0}$$

Type derivations

$$\dfrac{\dfrac{\texttt{1} : \textit{num} \qquad \texttt{2} : \textit{num}}{\texttt{\{+ 1 2\}} : \textit{num}} \qquad \texttt{3} : \textit{num}}{\texttt{\{+ \{+ 1 2\} 3\}} : \textit{num}}$$

- We can conclude that an expression has some type if we can come up with a derivation using the type rules.

Type derivations

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- We can conclude that an expression has some type if we can come up with a derivation using the type rules.
- Great, but given some expression, how can we find the right derivation, and its type
- And what if it doesn't have a type...

Finding a type

$$\Gamma \vdash \mathbf{e} : \tau$$

Let's try to find a type for this expression

`[] ⊢ {if true {+ 1 2} 3} : τ ?`

Finding a type

$$\Gamma \vdash \mathbf{e} : \tau$$

Let's try to find a type for this expression

`[]` $\vdash$ `{if true {+ 1 2} 3}` $: \tau$?

- What is the type of `{if true {+ 1 2} 3}`?
- Is there some τ that will satisfy the type judgment?

Finding a type

$$\Gamma \vdash \mathbf{e} : \tau$$

Let's try to find a type for this expression

`[] ⊢ {if true {+ 1 2} 3} : τ ?`

Let's try a rule:

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This one won't work (the expressions don't match)...

Finding a type

$$\Gamma \vdash \mathbf{e} : \tau$$

Let's try to find a type for this expression

$$[\,] \vdash \texttt{\{if true \{+ 1 2\} 3\}} : \tau \ ?$$

Try again:

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{bool} \qquad \Gamma \vdash \mathbf{e}_2 : \tau_0 \qquad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \texttt{\{if } \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\texttt{\}} : \tau_0}$$

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This works, but we still don't have a full derivation...

Finding a type

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Try again:

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So we have to try again...

Finding a type

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Let's try to find a type for this expression

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- In general, we are stuck doing an expensive search where we try every rule for every expression (with backtracking).

Finding a type

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Let’s try to find a type for this expression

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- In general, we are stuck doing an expensive search where we try every rule for every expression (with backtracking).
- But actually the type rules have some nice properties, so things aren’t really that difficult...

TFAE Rules

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Properties of TFAE types

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- There is only one rule that applies to any TFAE expression

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- There is only one rule that applies to any TFAE expression
- So there is only one (possible) type derivation for any expression

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- For any rule, Γ and $\mathbf{e}$ always determine τ

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- For any rule, Γ and $\mathbf{e}$ always determine τ
- Think of Γ and $\mathbf{e}$ as inputs — they give us the necessary information for recursive calls (premises).
- Think of τ as an output — the premises give us what we need to know to construct the result type

Properties of TFAE types

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- For any rule, Γ and **e** always determine τ
- Think of Γ and **e** as inputs — they give us the necessary information for recursive calls (premises).
- Think of τ as an output — the premises give us what we need to know to construct the result type
- So we can easily turn the type judgment into a *function*:

```
; type-check Γ e -> τ
;
```