Type Declarations

$$\Gamma \vdash \texttt{<num>} : \textit{num} \qquad [\ldots\ \texttt{<id>}\leftarrow\tau\ \ldots] \vdash \texttt{<id>} : \tau$$

$$\Gamma \vdash \texttt{true} : \textit{bool} \qquad \Gamma \vdash \texttt{false} : \textit{bool}$$

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{num} \qquad \Gamma \vdash \mathbf{e}_2 : \textit{num}}{\Gamma \vdash \{\texttt{+}\ \mathbf{e}_1\ \mathbf{e}_2\} : \textit{num}}$$

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{bool} \qquad \Gamma \vdash \mathbf{e}_2 : \tau_0 \qquad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \{\texttt{if}\ \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\} : \tau_0}$$

$$\frac{\Gamma[\texttt{<id>}\leftarrow\tau_1] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{\texttt{fun}\ \{\texttt{<id>} : \tau_1\}\ \mathbf{e}\} : (\tau_1 \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \rightarrow \tau_0) \qquad \Gamma \vdash \mathbf{e}_1 : \tau_1}{\Gamma \vdash \{\mathbf{e}_0\ \mathbf{e}_1\} : \tau_0}$$

Type Inference

$$\Gamma \vdash \texttt{<num>} : \textit{num} \qquad [\ldots\ \texttt{<id>}\leftarrow\tau\ \ldots] \vdash \texttt{<id>} : \tau$$

$$\Gamma \vdash \texttt{true} : \textit{bool} \qquad \Gamma \vdash \texttt{false} : \textit{bool}$$

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{num} \qquad \Gamma \vdash \mathbf{e}_2 : \textit{num}}{\Gamma \vdash \{\texttt{+}\ \mathbf{e}_1\ \mathbf{e}_2\} : \textit{num}}$$

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{bool} \qquad \Gamma \vdash \mathbf{e}_2 : \tau_0 \qquad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \{\texttt{if}\ \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\} : \tau_0}$$

$$\frac{\Gamma[\texttt{<id>}\leftarrow\tau_1] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{\texttt{fun}\ \{\texttt{<id>}\}\ \ \mathbf{e}\} : (\tau_1 \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \rightarrow \tau_0) \qquad \Gamma \vdash \mathbf{e}_1 : \tau_1}{\Gamma \vdash \{\mathbf{e}_0\ \mathbf{e}_1\} : \tau_0}$$

Do the rules still work? (Yes.)

$$\dfrac{\dfrac{[\texttt{x} \leftarrow \textit{num}]\ \texttt{x} : \textit{num} \qquad [\texttt{x} \leftarrow \textit{num}]\ \texttt{1} : \textit{num}}{[\texttt{x} \leftarrow \textit{num}]\ \{\texttt{+ x 1}\} : \textit{num}}}{\varnothing \vdash \{\texttt{fun}\ \{\texttt{x}\}\ \{\texttt{+ x 1}\}\} : (\textit{num}\ \texttt{->}\ \textit{num})}$$

But how do we implement them now?

Type Inference

- ***Type inference*** is the process of inserting type annotations where the programmer omits them

- We'll use explicit question marks, to make it clear where types are omitted

```
{fun {x : ?} {+ x 1}}
```

Type Inference

- ***Type inference*** is the process of inserting type annotations where the programmer omits them

- We'll use explicit question marks, to make it clear where types are omitted

```
{fun {x : ?} {+ x 1}}
```

```
<typeExpr>  ::=  num
             |   bool
             |   (<typeExpr> -> <typeExpr>)
             |   ?
```

Type Inference

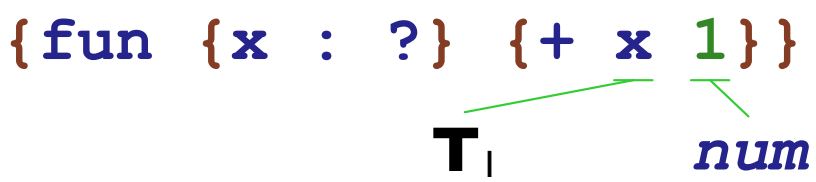
```
{fun {x : ?} {+ x 1}}
```

Type Inference

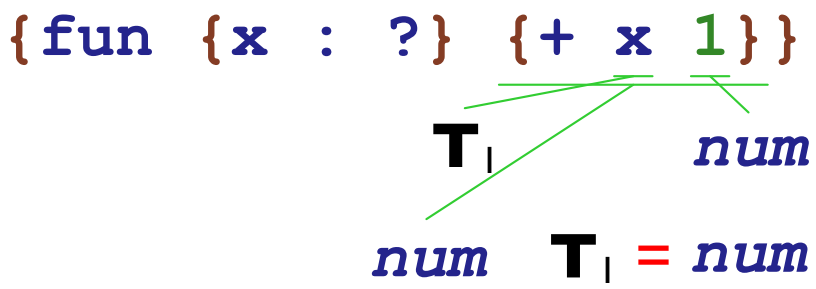
```
{fun {x : ?} {+ x 1}}
```

T_1

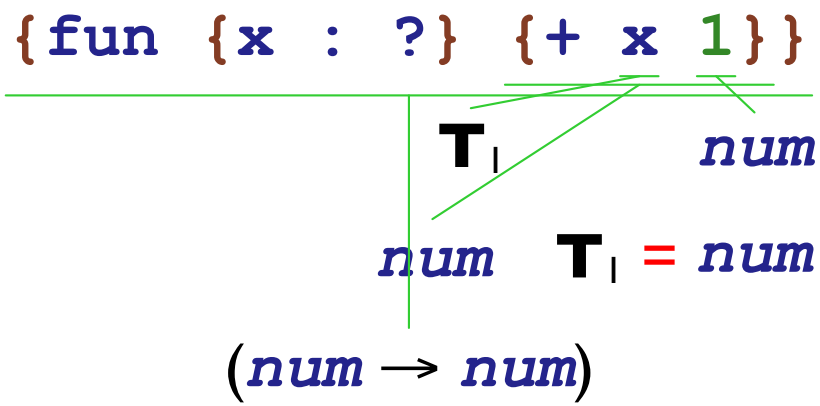
Type Inference



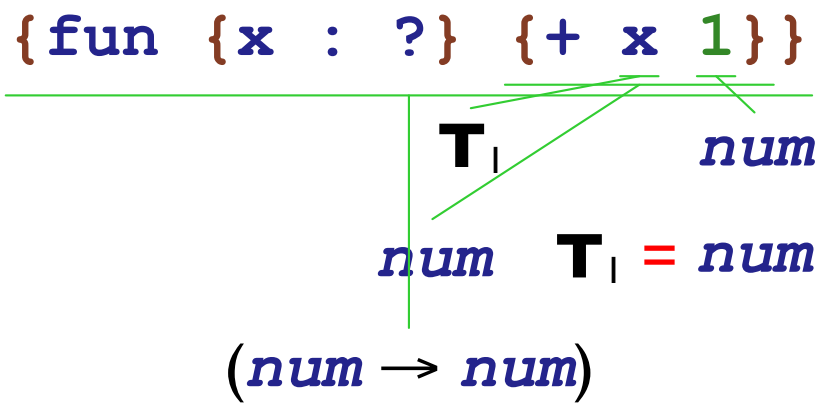
Type Inference



Type Inference

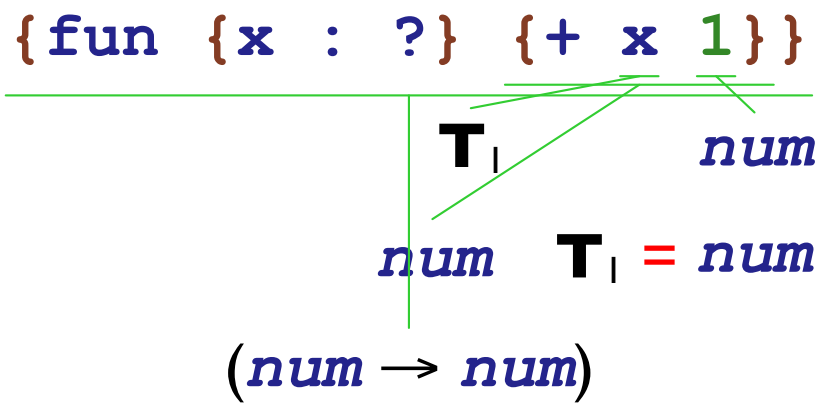


Type Inference



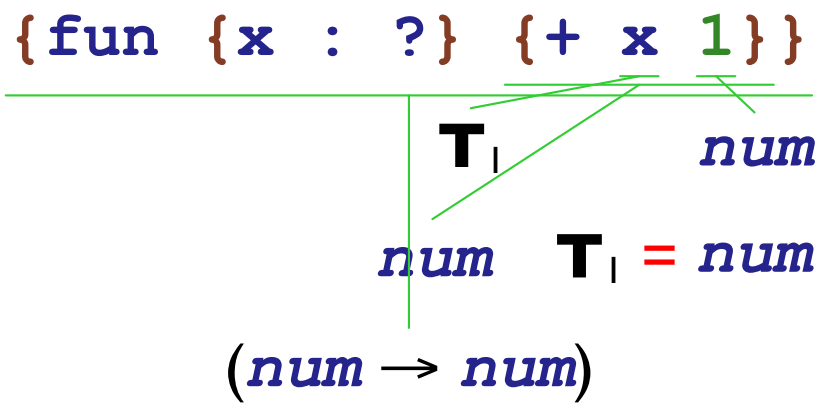
- Create a new type variable for each `?`
- Change type comparison to install type equivalences

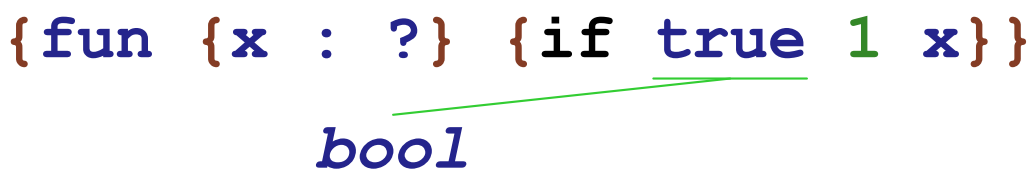
Type Inference



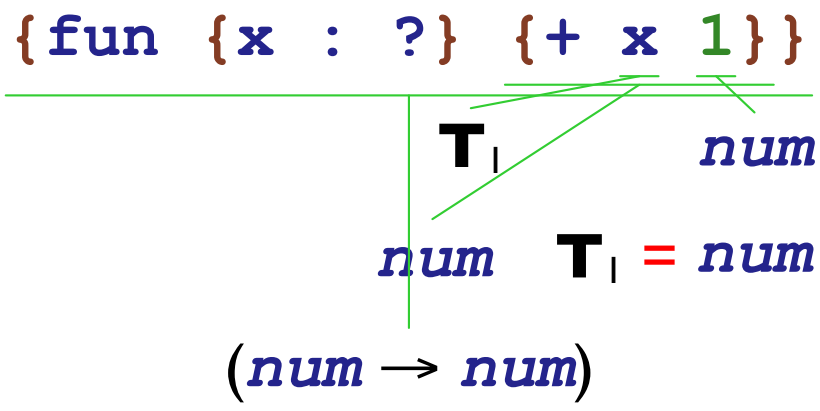

```
{fun {x : ?} {if true 1 x}}
```

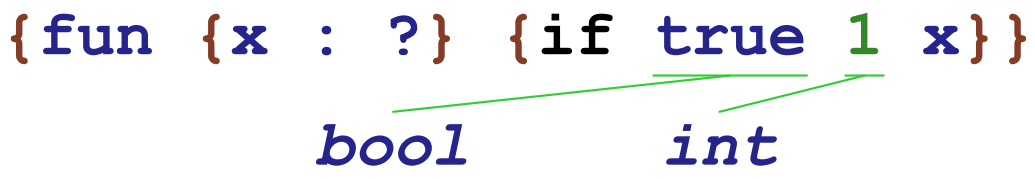
Type Inference



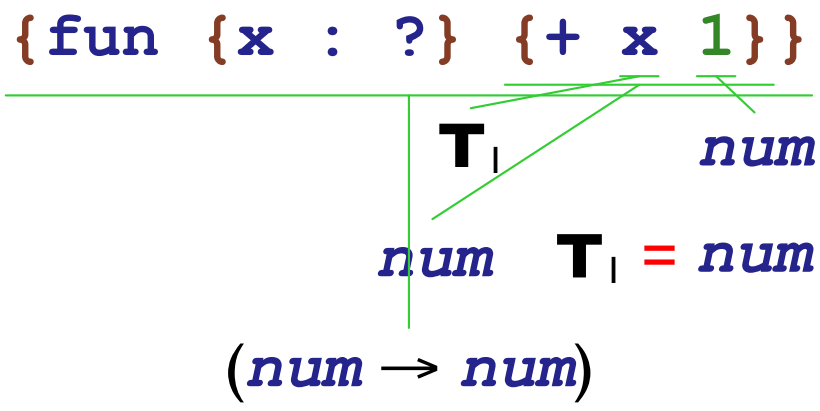


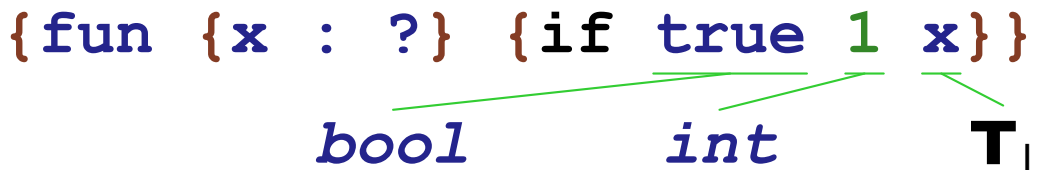
Type Inference



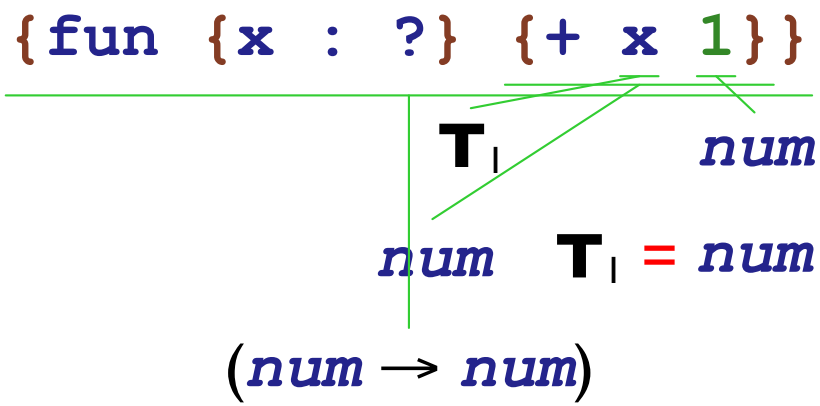


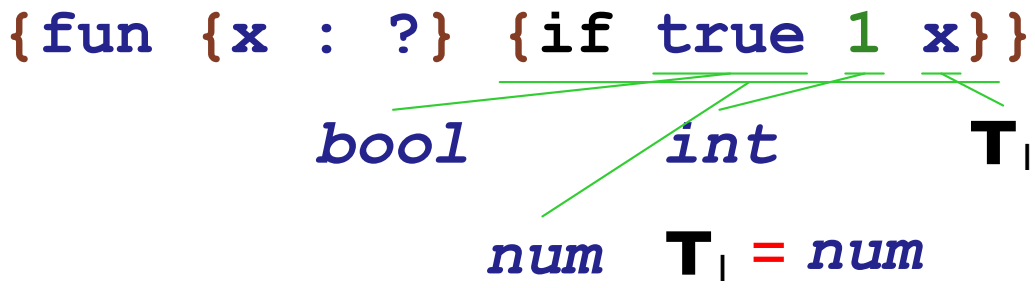
Type Inference



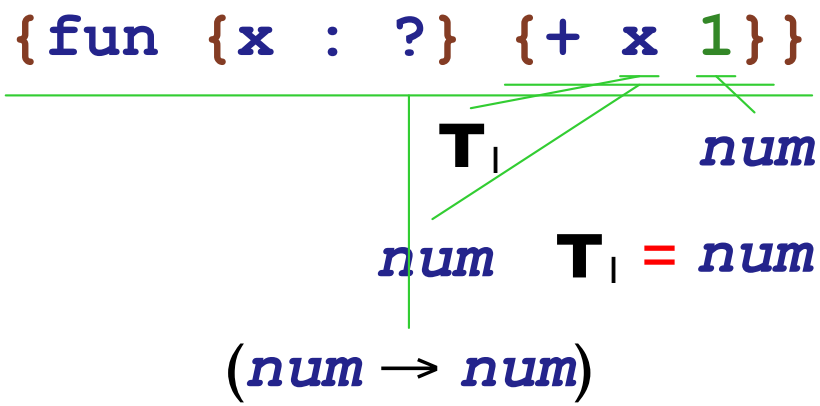


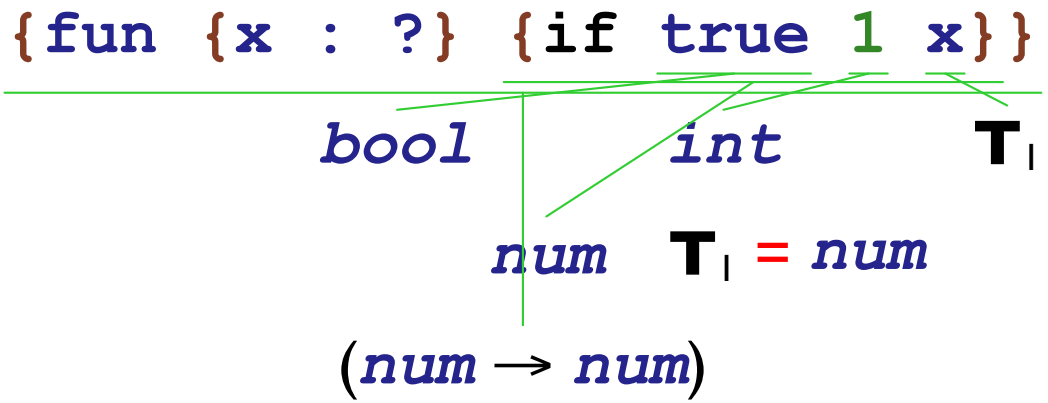
Type Inference





Type Inference





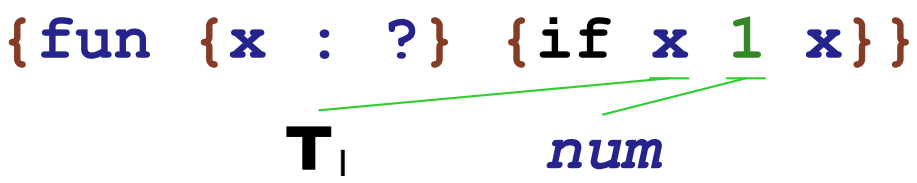
Type Inference: Impossible Cases

```
{fun {x : ?} {if x 1 x}}
```

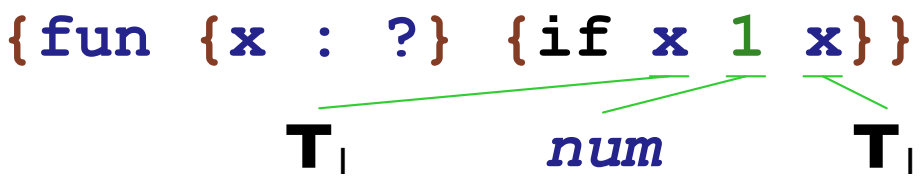
Type Inference: Impossible Cases

```
{fun {x : ?} {if x 1 x}}
```

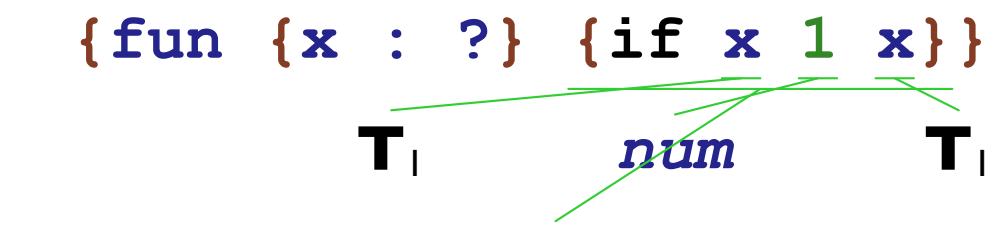
Type Inference: Impossible Cases



Type Inference: Impossible Cases



Type Inference: Impossible Cases



no type: **T**| can't be both ***bool*** and ***num***

Type Inference: Many Cases

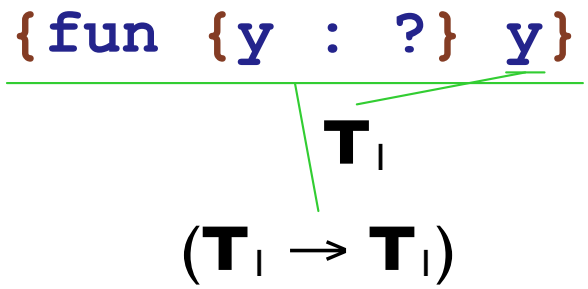
```
{fun {y : ?} y}
```

Type Inference: Many Cases

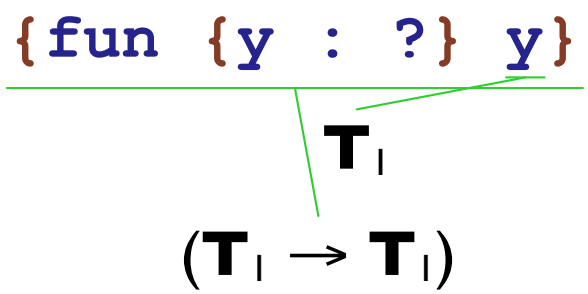
```
{fun {y : ?} y}
```

T_1

Type Inference: Many Cases



Type Inference: Many Cases



- Sometimes, more than one type works
  - (*num* → *num*)
  - (*bool* → *bool*)
  - ((*num* → *bool*) → (*num* → *bool*))

  so the type checker leaves variables in the reported type

Type Inference: Function Calls

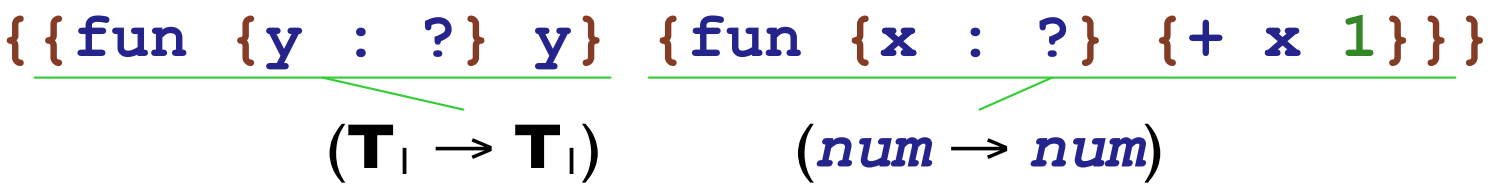
```
{{fun {y : ?} y} {fun {x : ?} {+ x 1}}}
```

Type Inference: Function Calls

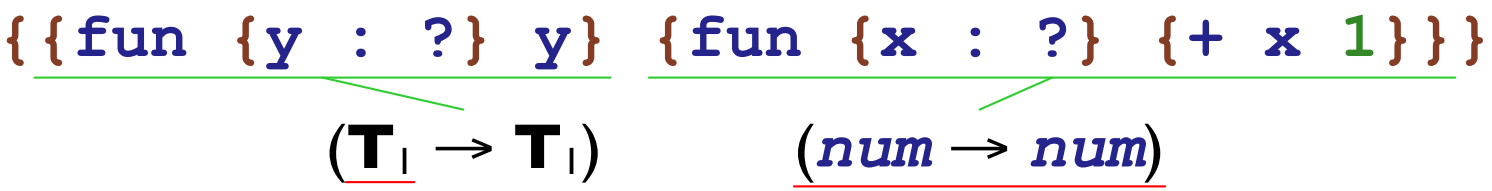
```
{{fun {y : ?} y} {fun {x : ?} {+ x 1}}}
```

$(\mathbf{T}_1 \rightarrow \mathbf{T}_1)$

Type Inference: Function Calls

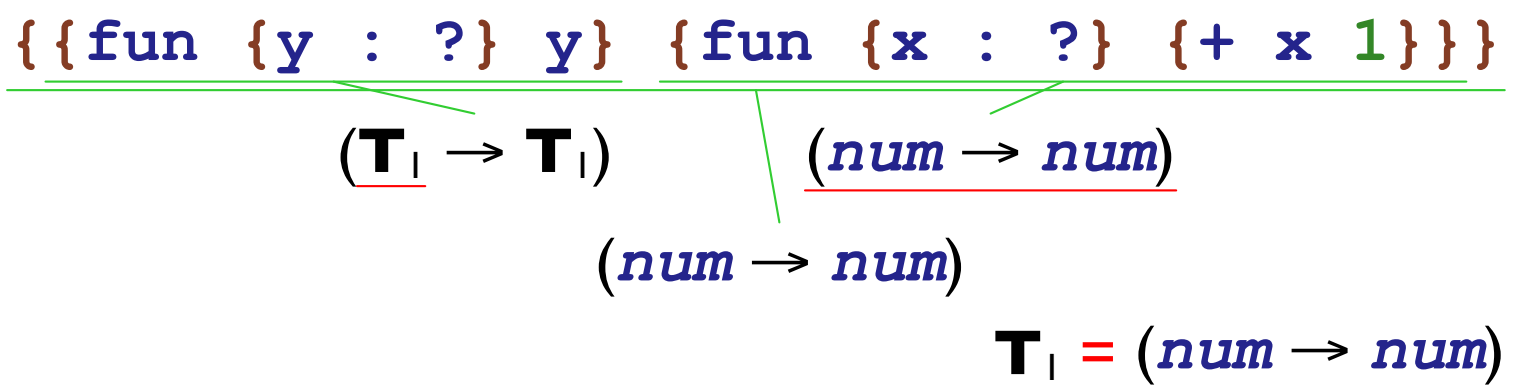


Type Inference: Function Calls



$\mathbf{T}_1 = (num \rightarrow num)$

Type Inference: Function Calls



Type Inference: Function Calls

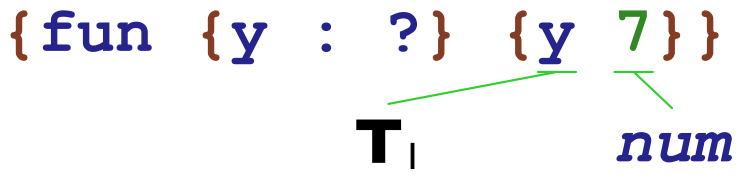
```
{fun {y : ?} {y 7}}
```

Type Inference: Function Calls

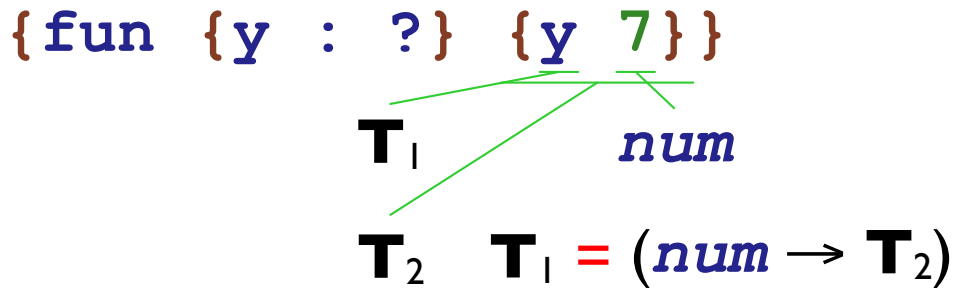
```
{fun {y : ?} {y 7}}
```

T_1

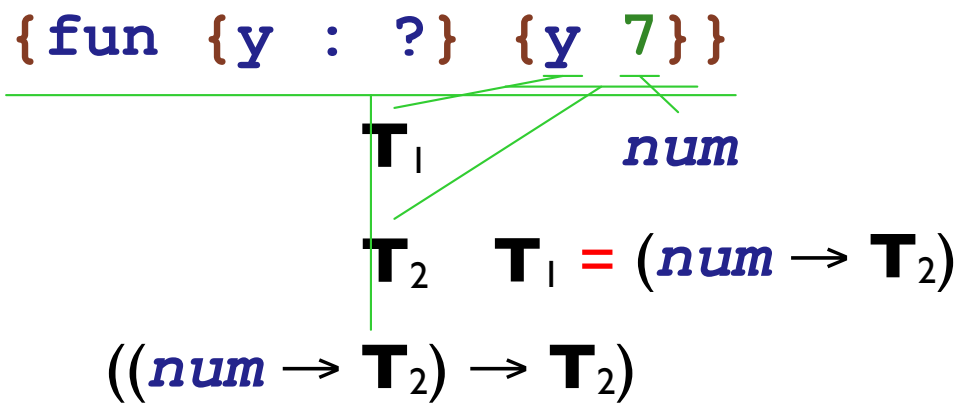
Type Inference: Function Calls



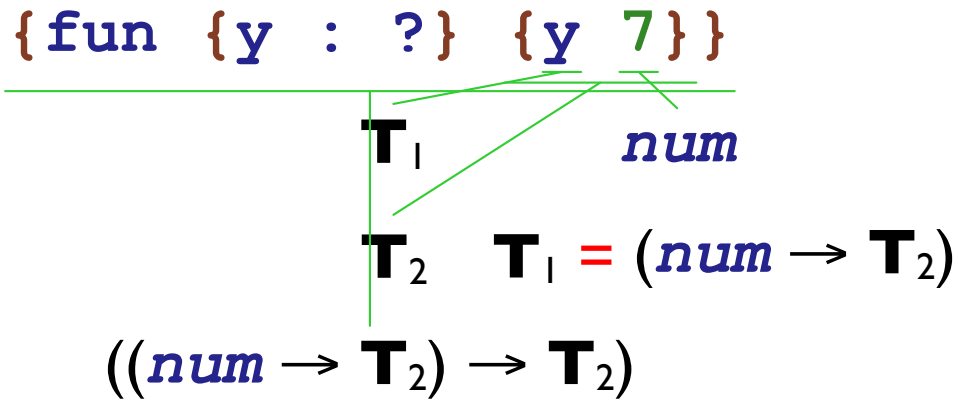
Type Inference: Function Calls



Type Inference: Function Calls



Type Inference: Function Calls



- In general, create a new type variable record for the result of a function call

Type Inference: Cyclic Equations

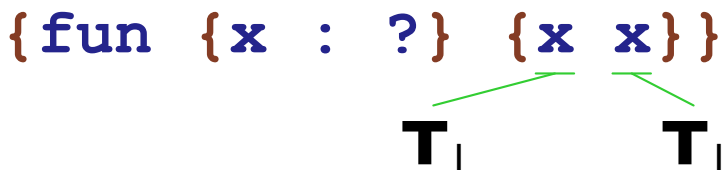
```
{fun {x : ?} {x x}}
```

Type Inference: Cyclic Equations

```
{fun {x : ?} {x x}}
```

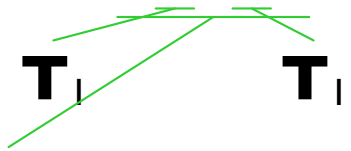
T_1

Type Inference: Cyclic Equations



Type Inference: Cyclic Equations

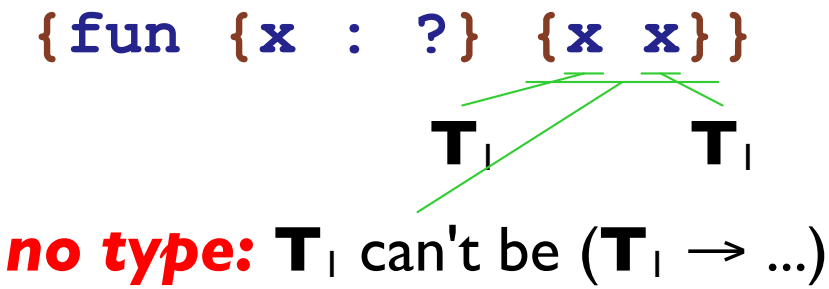
```
{fun {x : ?} {x x}}
```



no type: T_1 can't be ($T_1 \rightarrow$ ...)

- T_1 can't be ***int***
- T_1 can't be ***bool***
- Suppose T_1 is ($T_2 \rightarrow T_3$)
  - T_2 must be T_1
  - So we won't get anywhere!

Type Inference: Cyclic Equations



The ***occurs check***:

- When installing a type equivalence, make sure that the new type for **T** doesn't already contain **T**

Type Unification

Unify a type variable **T** with a type τ_2:

- If **T** is set to τ_1, unify τ_1 and τ_2
- If τ_2 is already equivalent to **T**, succeed
- If τ_2 contains **T**, then fail
- Otherwise, set **T** to τ_2 and succeed

Unify a type τ_1 to type τ_2:

- If τ_2 is a type variable **T**, then unify **T** and τ_1
- If τ_1 and τ_2 are both ***num*** or ***bool***, succeed
- If τ_1 is $(\tau_3 \rightarrow \tau_4)$ and τ_2 is $(\tau_5 \rightarrow \tau_6)$, then
  - unify τ_3 with τ_5
  - unify τ_4 with τ_6
- Otherwise, fail

TIFAE Grammar

```
<TIFAE>  ::=  <num>
           |  {+ <TIFAE> <TIFAE>}
           |  {- <TIFAE> <TIFAE>}
           |  <id>
           |  {fun {<id> : <TE>} <TIFAE>}
           |  {<TIFAE> <TIFAE>}
           |  {if0 <TIFAE> <TIFAE> <TIFAE>}
           |  {rec {<id> : <TE> <TIFAE>} <TIFAE>}

<TE>     ::=  num
           |  (<TE> -> <TE>)
           |  ?
```

Representing Expressions

```
(define-type TFAE
  [num (n : number)]
  [add (l : TFAE)
       (r : TFAE)]
  [sub (l : TFAE)
       (r : TFAE)]
  [id (name : symbol)]
  [fun (name : symbol)
       (t : TE)   ; different
       (body : TFAE)]
  [app (rator : TFAE)
       (rand : TFAE)])
```

Representing Type Variables

```
(define-type Type
  [numT]
  [boolT]
  [arrowT (arg : Type)
          (result : Type)]
  [varT (is : (boxof MaybeType))])

(define-type TE
  [numTE]
  [boolTE]
  [arrowTE
   (arg : TE)
   (result : TE)]
  [guessTE])

(define-type MaybeType
  [none]
  [some (t : Type)])
```

Parsing Types

```
(define parse-type : (TE -> Type)
  (lambda (te)
    (type-case TE te
      [numTE () (numT)]
      [boolTE () (boolT)]
      [arrowTE (d r)
               (arrowT (parse-type d)
                       (parse-type r))]
      [guessTE () (varT (box (none)))])))
```

Type Unification

```
(define unify! : (Type Type -> ())
  (lambda (t1 t2)
    (type-case Type t1
      [varT (b) (type-case MaybeType (unbox b)
                  [some (t1-2) (unify! t1-2 t2)]
                  [none ()
                        (type-case Type t2
                          [varT (b2)
                                (type-case MaybeType (unbox b2)
                                  [some (t2-2) (unify! t1 t2-2)]
                                  [none () (if (eq? b b2)
                                               (values)
                                               (begin
                                                 (set-box!
                                                  b
                                                  (some t2))
                                                 (values))))])]
                          [else ...])])]
      [numT () ...]
      [boolT () ...]
      [arrowT (d1 r1) ...])))
```

Type Unification

```
(define unify! : (Type Type -> ())
  (lambda (t1 t2)
    (type-case Type t1
      [varT (b) (type-case MaybeType (unbox b)
                  [some (t1-2) (unify! t1-2 t2)]
                  [none ()
                        (type-case Type t2
                          [varT (b2)
                                ...]
                          [else (if (occurs? b t2)
                                    (error 'type-check "failed")
                                    (begin
                                      (set-box! b (some t2))
                                      (values)))])])]
      [numT () ...]
      [boolT () ...]
      [arrowT (d1 r1) ...])))
```

Type Unification

```
(define unify! : (Type Type -> ())
  (lambda (t1 t2)
    (type-case Type t1
      [varT (b) ...]
      [numT () (type-case Type t2
                 [varT (b) (unify! t2 t1)]
                 [numT () (values)]
                 [else (error 'type-check "failed")])]
      [boolT () (type-case Type t2
                  [varT (b) (unify! t2 t1)]
                  [boolT () (values)]
                  [else (error 'type-check "failed")])]
      [arrowT (d1 r1) (type-case Type t2
                        [varT (b) (unify! t2 t1)]
                        [arrowT (d2 r2)
                                (begin (unify! d1 d2)
                                       (unify! r1 r2))]
                        [else (error 'type-check "failed")])])))
```

Occurs Check

```
(define occurs? : ((boxof MaybeType) Type -> boolean)
  (lambda (b t)
    (type-case Type t
      [varT (b2) (or (eq? b b2)
                     (type-case MaybeType (unbox b2)
                       [none () false]
                       [some (t2-2) (occurs? b t2-2)]))]
      [numT () false]
      [arrowT (d r) (or (occurs? b d)
                        (occurs? b r))])))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [num (n) (numT)]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [add (l r)
           ... (typecheck l env) ...
           ... (typecheck r env) ...]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [add (l r) (begin
                   (unify! (typecheck l env) (numT) l)
                   (unify! (typecheck r env) (numT) r)
                   (numT))]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [id (name) (get-type name env)]
      [fun (name te body)
           (local [(define arg-type (parse-type te))]
             (arrowT arg-type
                     (typecheck body (aBind name
                                            arg-type
                                            env))))]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [app (fn arg)
           ... (typecheck fn env) ...
           ... (typecheck arg env) ...]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [app (fn arg)
           (local [(define result-type (varT (box (none))))]
             ... (arrowT (typecheck arg env)
                         result-type)
             ... (typecheck fn env) ...)]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [app (fn arg)
           (local [(define result-type (varT (box (none))))]
             (begin
               (unify! (arrowT (typecheck arg env)
                               result-type)
                       (typecheck fn env)
                       fn)
               result-type))]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [if0 (test-expr then-expr else-expr)
           ... (typecheck test-expr env) ...
           ... (typecheck then-expr env) ...
           ... (typecheck else-expr env) ...]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [if0 (test-expr then-expr else-expr)
           (begin
             (unify! (typecheck test-expr env) (numT) test-expr)
             ... (typecheck then-expr env) ...
             ... (typecheck else-expr env) ...)]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [if0 (test-expr then-expr else-expr)
           (begin
             (unify! (typecheck test-expr env) (numT) test-expr)
             (local [(define test-ty
                       (typecheck then-expr env))]
               (begin
                 (unify! test-ty
                         (typecheck else-expr env)
                         else-expr)
                 test-ty)))]
      ...)))
```

TIFAE Type Checker

```
(define typecheck : (FAE TypeEnv -> Type)
  (lambda (fae env)
    (type-case FAE fae
      ...
      [rec (name ty rhs-expr body-expr)
           (local [(define rhs-ty (parse-type ty))
                   (define new-ds (aBind name
                                         rhs-ty
                                         env))]
             (begin
               (unify! rhs-ty (typecheck rhs-expr new-ds) rhs-expr)
               (typecheck body-expr new-ds)))]
      ...)))
```