

Inference rules

$$\frac{A \quad B}{C}$$

This is a *rule*, which says: If I know *A* and *B*, then I can conclude *C* (could be more than just two things above the bar)

Terminology: *A* and *B* are *premises* *C* is the *conclusion*

Inference Rule Example: Logical Or

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

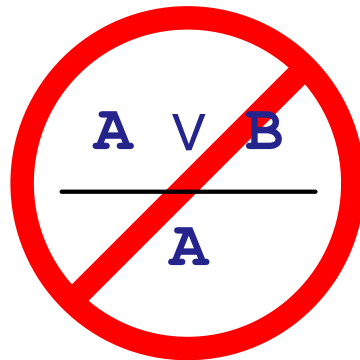
$$\frac{A \vee B}{A}$$

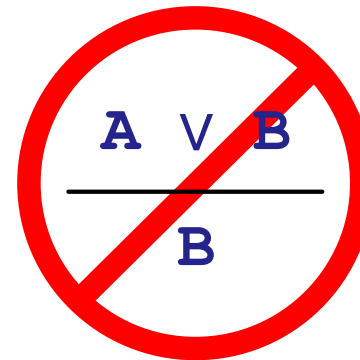
$$\frac{A \vee B}{B}$$

Inference Rule Example: Logical Or

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$


$$\frac{A \vee B}{A}$$


$$\frac{A \vee B}{B}$$

Type Relation

We are defining a *type judgment*

$$\Gamma \vdash \mathbf{e} : \tau$$

Which means: with the type bindings in Γ , I can conclude that \mathbf{e} has the type τ

Γ is the *type environment*: a map from $\langle \mathbf{id} \rangle$ to τ (type)

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{num}$

$\Gamma \vdash \text{true} : \text{bool}$

$\Gamma \vdash \text{false} : \text{bool}$

$$\frac{\Gamma \vdash \mathbf{e}_1 : \text{num} \quad \Gamma \vdash \mathbf{e}_2 : \text{num}}{\Gamma \vdash \{+ \mathbf{e}_1 \ \mathbf{e}_2\} : \text{num}}$$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{num}$

$\Gamma \vdash \text{true} : \text{bool}$

$\Gamma \vdash \text{false} : \text{bool}$

$$\frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash \{+ e_1 e_2\} : \text{num}}$$

$1 : \text{num}$

$\text{true} : \text{bool}$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{num}$

$\Gamma \vdash \text{true} : \text{bool}$

$\Gamma \vdash \text{false} : \text{bool}$

$$\frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash \{+ e_1 e_2\} : \text{num}}$$

$1 : \text{num}$

$\text{true} : \text{bool}$

$$\frac{1 : \text{num} \quad 2 : \text{num}}{\{+ 1 2\} : \text{num}}$$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{num}$

$\Gamma \vdash \text{true} : \text{bool}$

$\Gamma \vdash \text{false} : \text{bool}$

$\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}$

$\Gamma \vdash \{+ e_1 e_2\} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num} \quad 2 : \text{num}$

$\{+ 1 2\} : \text{num}$

$1 : \text{num} \quad \text{false} : \text{bool}$

$\{+ 1 \text{false}\} : \text{no type}$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{num}$

$\Gamma \vdash \text{true} : \text{bool}$

$\Gamma \vdash \text{false} : \text{bool}$

$$\frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash \{+ e_1 e_2\} : \text{num}}$$
$$\frac{\frac{1 : \text{num} \quad 2 : \text{num}}{\{+ 1 2\} : \text{num}} \quad 3 : \text{num}}{\{+ \{+ 1 2\} 3\} : \text{num}}$$

Types: Conditionals

```
{if true 1 2}
```

Types: Conditionals

```
bool {if true 1 2}
```

Types: Conditionals

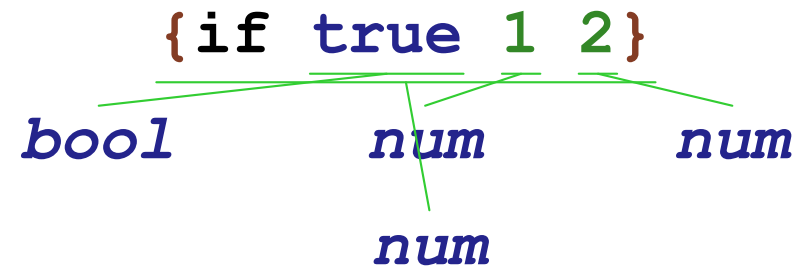
```
{if true 1 2}
```

bool *num*

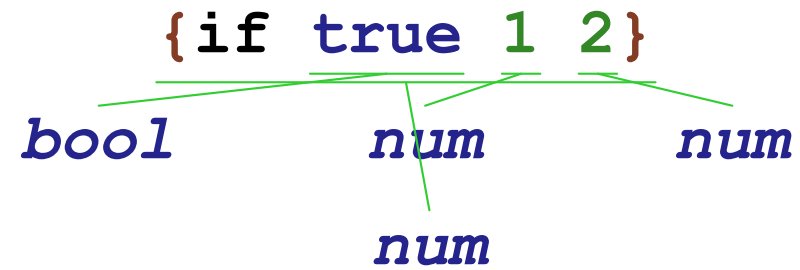
Types: Conditionals



Types: Conditionals



Types: Conditionals



`{if {+ 1 2} 1 2}`

Types: Conditionals

```
{if true 1 2}
```

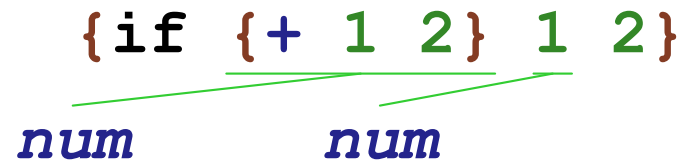
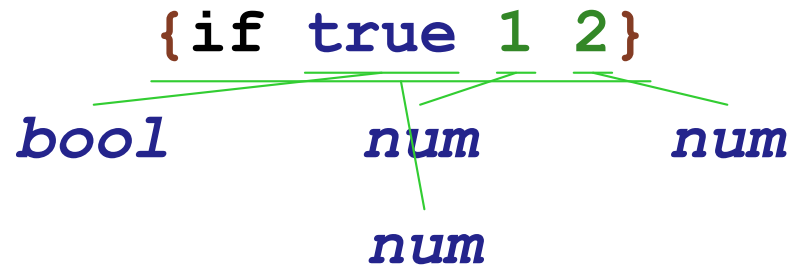
bool *num* *num*

num

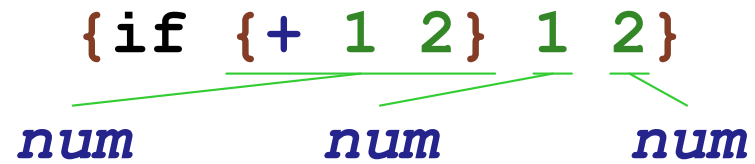
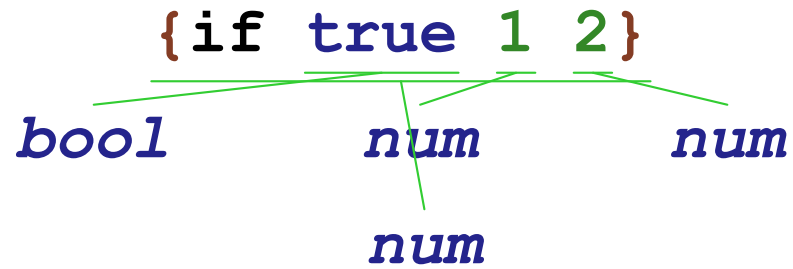
```
{if {+ 1 2} 1 2}
```

num

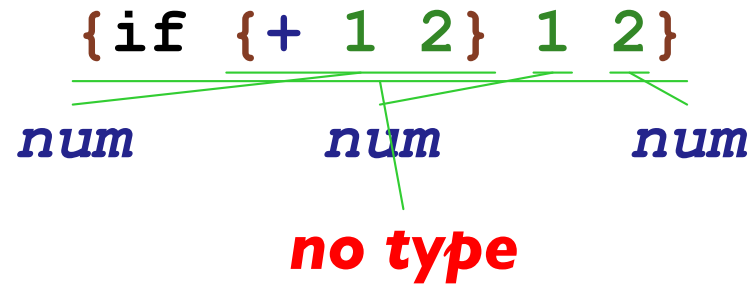
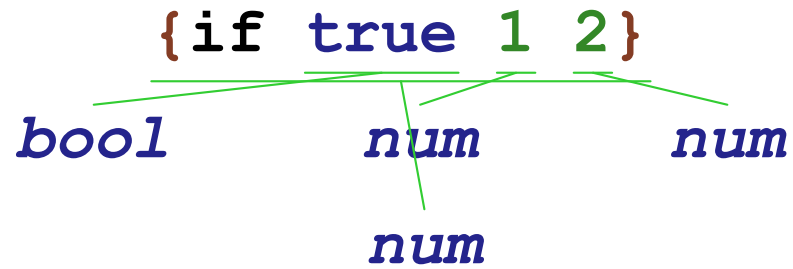
Types: Conditionals



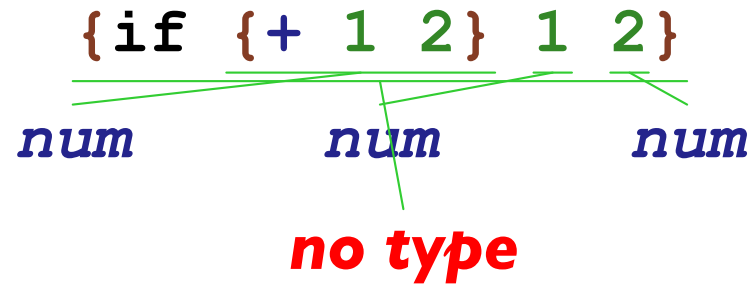
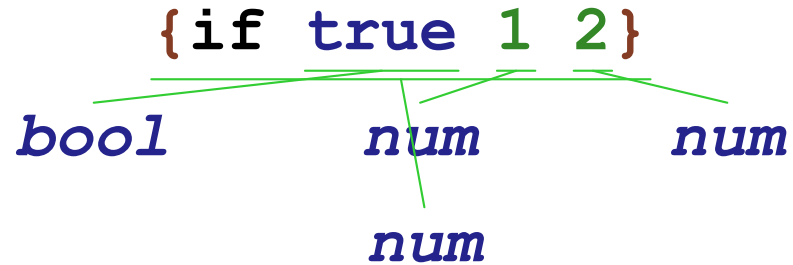
Types: Conditionals



Types: Conditionals

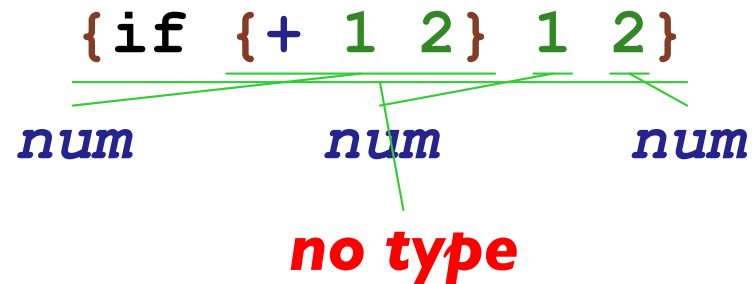
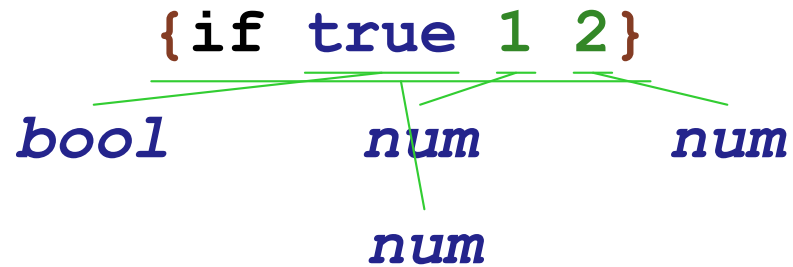


Types: Conditionals

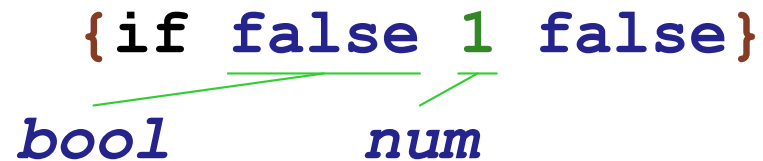
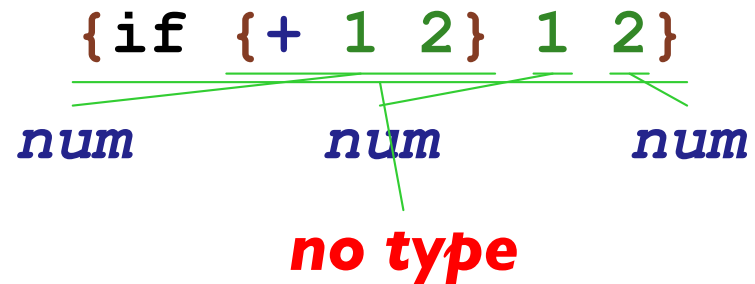
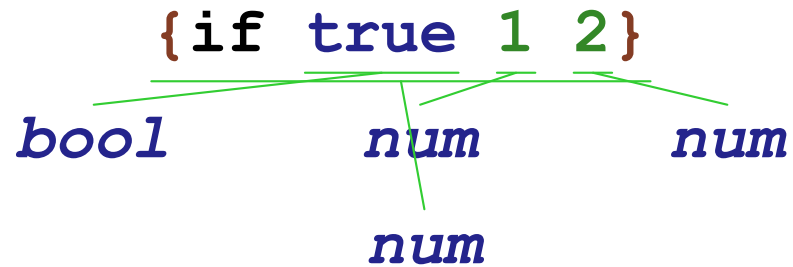


`{if false 1 false}`

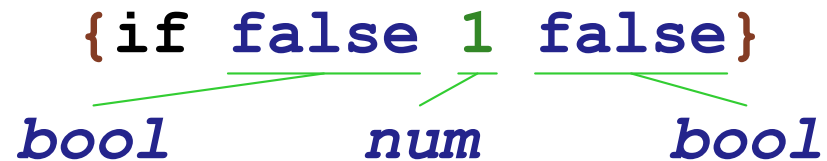
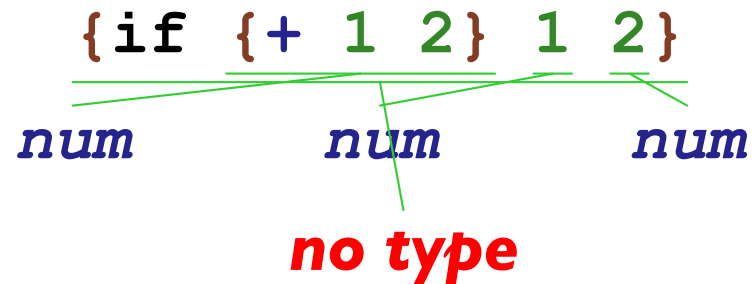
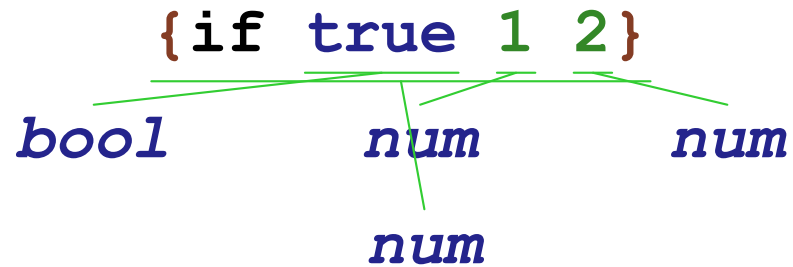
Types: Conditionals



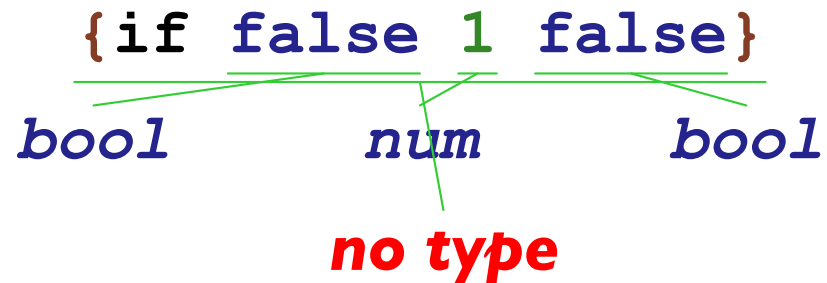
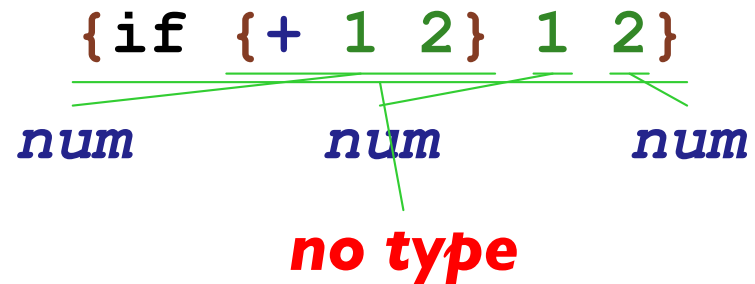
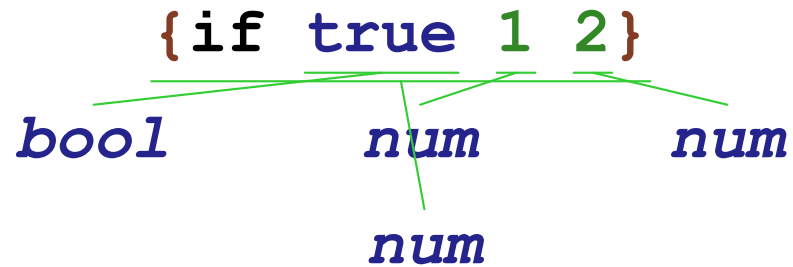
Types: Conditionals



Types: Conditionals



Types: Conditionals



Conditional Type Rules

$$\frac{\Gamma \vdash \mathbf{e}_1 : \mathit{bool} \quad \Gamma \vdash \mathbf{e}_2 : \tau_0 \quad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \{\mathbf{if} \ \mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3\} : \tau_0}$$

Conditional Type Rules

$$\frac{\Gamma \vdash e_1 : \mathit{bool} \quad \Gamma \vdash e_2 : \tau_0 \quad \Gamma \vdash e_3 : \tau_0}{\Gamma \vdash \{\mathit{if } e_1 e_2 e_3\} : \tau_0}$$

$$\frac{\mathit{true} : \mathit{bool} \quad 1 : \mathit{num} \quad 2 : \mathit{num}}{\{\mathit{if } \mathit{true } 1 2\} : \mathit{num}}$$

Conditional Type Rules

$$\Gamma \vdash e_1 : \mathit{bool} \quad \Gamma \vdash e_2 : \tau_0 \quad \Gamma \vdash e_3 : \tau_0$$

$$\Gamma \vdash \{\mathbf{if} \ e_1 \ e_2 \ e_3\} : \tau_0$$
$$\mathit{true} : \mathit{bool} \quad 1 : \mathit{num} \quad 2 : \mathit{num}$$

$$\{\mathbf{if} \ \mathit{true} \ 1 \ 2\} : \mathit{num}$$
$$\{+ \ 1 \ 2\} : \mathit{num} \quad 1 : \mathit{num} \quad 2 : \mathit{num}$$

$$\{\mathbf{if} \ \{+ \ 1 \ 2\} \ 1 \ 2\} : \mathbf{no \ type}$$

Conditional Type Rules

$$\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau_0 \quad \Gamma \vdash e_3 : \tau_0$$

$$\Gamma \vdash \{\text{if } e_1 \ e_2 \ e_3\} : \tau_0$$
$$\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}$$

$$\{\text{if true } 1 \ 2\} : \text{num}$$
$$\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}$$

$$\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}$$
$$\text{false} : \text{bool} \quad 1 : \text{num} \quad \text{false} : \text{bool}$$

$$\{\text{if false } 1 \ \text{false}\} : \text{no type}$$

Types: Variables and Functions

x : **no type**


Types: Variables and Functions

`x : no type`

`{fun {x : bool} x}`

Types: Variables and Functions

x : **no type**

```
{fun {x : bool} x}
      
      bool
```

Types: Variables and Functions

x : **no type**

`{fun {x : bool} x}`

`bool`

`(bool → bool)`

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

(bool → bool)

`{fun {x : bool} {if x 1 2}}`

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

(bool → bool)

`{fun {x : bool} {if x 1 2}}`

bool

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

bool

num

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

bool

num

num

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

$(bool \rightarrow bool)$

`{fun {x : bool} {if x 1 2}}`

bool

num

num

num

Types: Variables and Functions

x : no type

`{fun {x : bool} x}`

bool

(bool → bool)

`{fun {x : bool} {if x 1 2}}`

bool

num

num

num

(bool → num)

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \mathbf{x} \} : (\text{bool} \rightarrow \text{bool})$$

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \mathbf{x} \} : (\text{bool} \rightarrow \text{bool})$$
$$[\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{x} : \text{bool} \quad [\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{1} : \text{num} \quad [\mathbf{x} \leftarrow \text{bool}] \vdash \mathbf{2} : \text{num}$$

$$[\mathbf{x} \leftarrow \text{bool}] \vdash \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} : \text{num}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{bool} \} \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} \} : (\text{bool} \rightarrow \text{num})$$

Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}
```

Types: Function Calls

```
{{fun {x : bool} {if x 1 2}} true}  
  (bool → num)
```

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}
(bool → num) *bool*

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

(bool → num) *bool*

num

Types: Function Calls

$$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$$

`{{fun {x : bool} {if x 1 2}} 5}`

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}
(bool → num) bool
num

{{fun {x : bool} {if x 1 2}} 5}
(bool → num)

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool}}$
 num

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

(bool → num) bool

num

{{fun {x : bool} {if x 1 2}} 5}

(bool → num) num

no type

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

(bool → num) bool

num

{{fun {x : bool} {if x 1 2}} 5}

(bool → num) num

no type

{7 5}

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

(bool → num) bool

num

{{fun {x : bool} {if x 1 2}} 5}

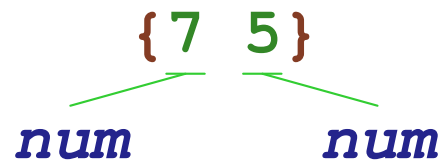
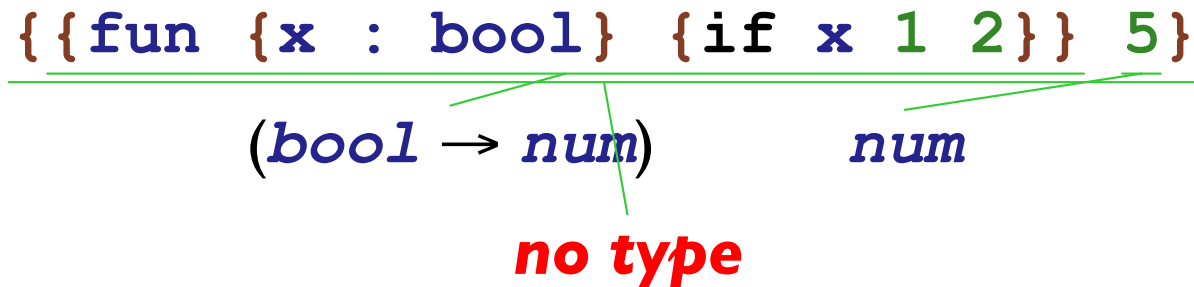
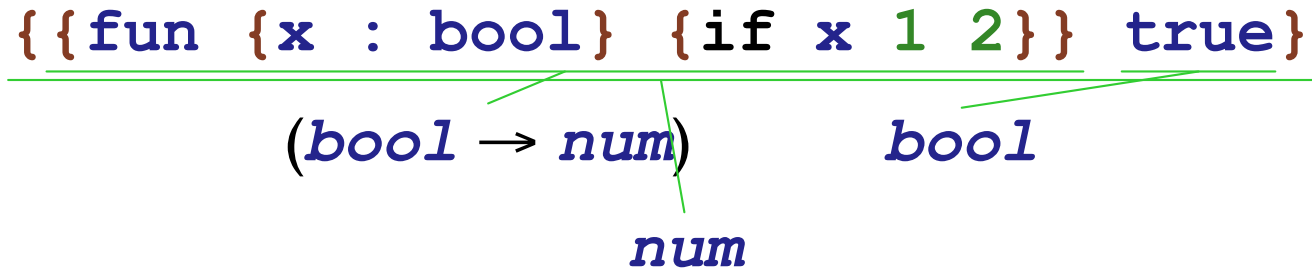
(bool → num) num

no type

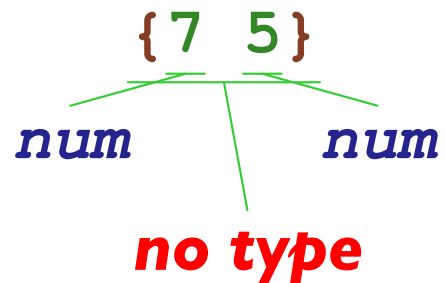
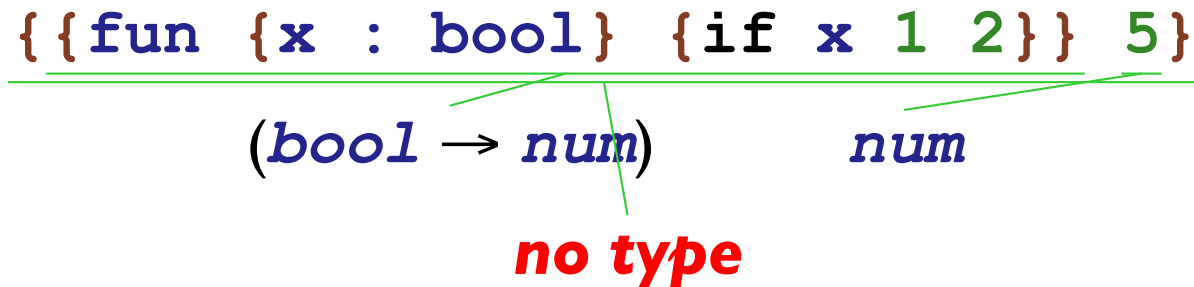
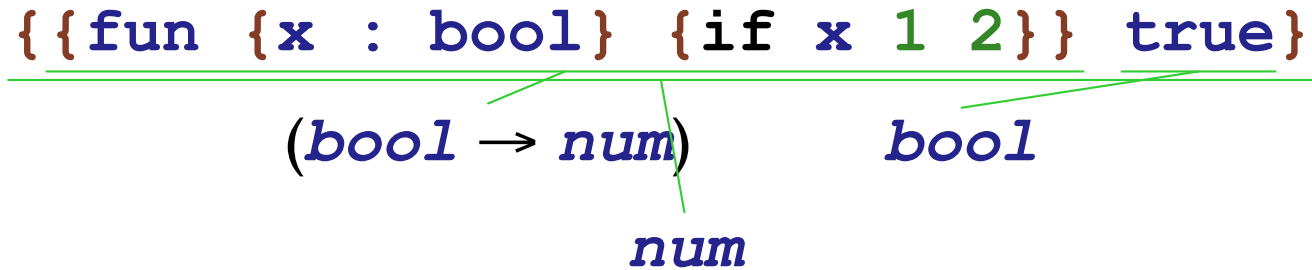
{7 5}

num

Types: Function Calls



Types: Function Calls



Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{\mathbf{x} : \text{bool}\} \{\text{if } \mathbf{x} \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathbf{if} \ \mathbf{x} \ 1 \ 2\}\} : (\mathit{bool} \rightarrow \mathit{num}) \quad \emptyset \vdash \mathit{true} : \mathit{bool}}{\emptyset \vdash \{\{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathbf{if} \ \mathbf{x} \ 1 \ 2\}\} \ \mathit{true}\} : \mathit{num}}$$

$$\frac{\emptyset \vdash \{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathbf{if} \ \mathbf{x} \ 1 \ 2\}\} : (\mathit{bool} \rightarrow \mathit{num}) \quad \emptyset \vdash 5 : \mathit{num}}{\emptyset \vdash \{\{\mathbf{fun} \ \{\mathbf{x} : \mathit{bool}\} \ \{\mathbf{if} \ \mathbf{x} \ 1 \ 2\}\} \ 5\} : \mathbf{no \ type}}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{num} \quad \emptyset \vdash 5 : \text{num}}$$

$$\emptyset \vdash \{7 \ 5\} : \text{no type}$$


Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y} }
```

Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y}}
```

num



Types: Multiple Arguments

```
{ fun {x : num y : num} {+ x y}}
```

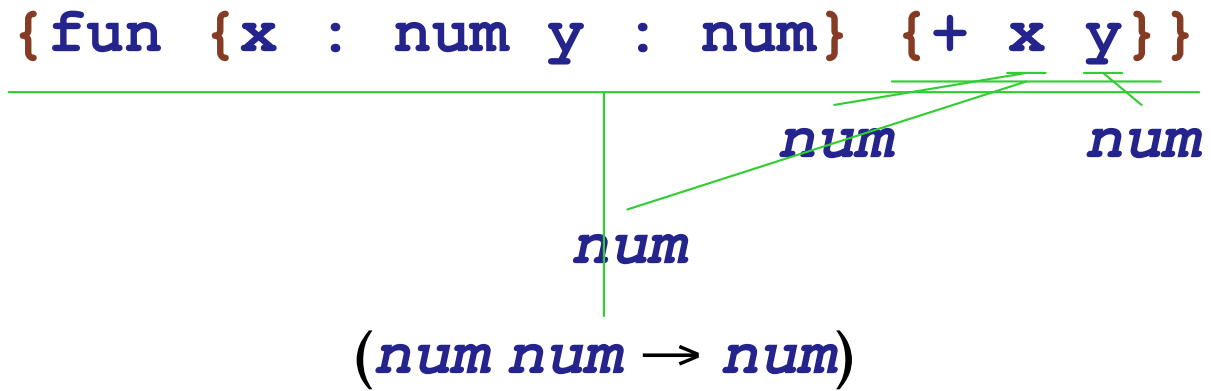
num *num*

Types: Multiple Arguments

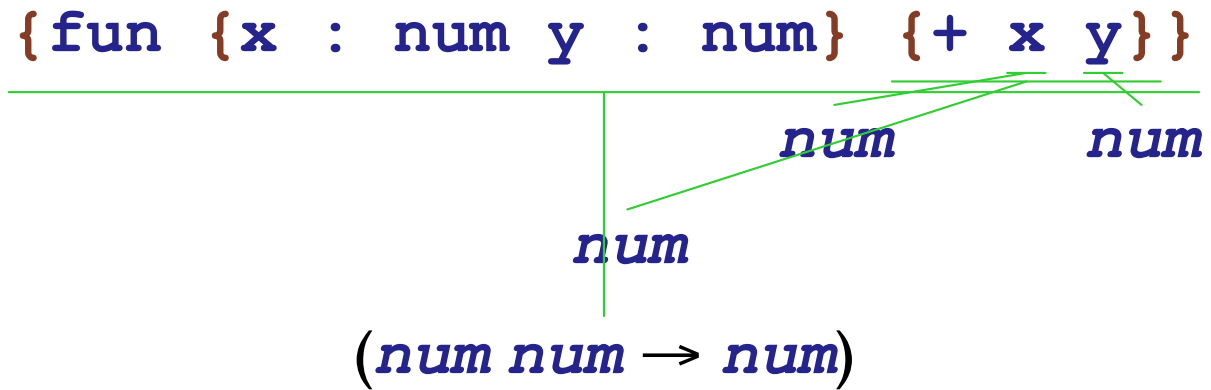
```
{ fun {x : num y : num} {+ x y}}
```

num *num* *num*

Types: Multiple Arguments



Types: Multiple Arguments



`{{fun {x : num y : num} {+ x y}} 5 6}`

Types: Multiple Arguments

$\frac{\text{\{fun \{x : num y : num\} \{+ x y\}\}}{\text{\{num num \rightarrow num\}}$

$\frac{\text{\{\{fun \{x : num y : num\} \{+ x y\}\} 5 6\}}{\text{\{num num \rightarrow num\}}$

Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

num *num*

num

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

$(num\ num \rightarrow num)$ *num*

Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

num *num*

num

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

$(num\ num \rightarrow num)$ *num* *num*

Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

num

Types: Multiple Arguments

`{fun {x : num y : num} {+ x y}}`

num *num*

num

$(num\ num \rightarrow num)$

`{{fun {x : num y : num} {+ x y}} 5 6}`

$(num\ num \rightarrow num)$ *num* *num*

num

`{{fun {x : num y : num} {+ x y}} 5}`

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num} \quad \text{num}}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}}$
 num

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num})}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num} \quad \text{num}}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}}$
 num

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$

Types: Multiple Arguments

$\frac{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}}{\text{num} \quad \text{num}} \text{num}$
 $(\text{num num} \rightarrow \text{num})$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num} \quad \text{num}} \text{num}$

$\frac{\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}}{(\text{num num} \rightarrow \text{num}) \quad \text{num}}$
no type

Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_l \leftarrow \tau_l \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{\text{fun } \{\langle \text{id} \rangle_l : \tau_l \dots \langle \text{id} \rangle_n : \tau_n\} \mathbf{e}\} : (\tau_l \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_l \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \mathbf{e}_l : \tau_l \quad \dots \quad \Gamma \vdash \mathbf{e}_n : \tau_n}{\Gamma \vdash \{\mathbf{e}_0 \ \mathbf{e}_l \ \dots \ \mathbf{e}_n\} : \tau_0}$$