

Typing Rules

What are type systems made of?

- Type systems are formal systems
- Usually described using logical rules
 - Provide a *specification* for what is well-formed
 - Rather than an *implementation* for determining whether something is well-formed
- If we want a *type checker* we turn these rules into code
 - Can be easy or hard, depending on the type system

Type Relation

We are defining a *type judgment*

$$\Gamma \vdash \mathbf{e} : \tau$$

Which means: with the type bindings in Γ , I can conclude that $\mathbf{e}$ has the type τ

Which implies that $\mathbf{e}$ is well-formed

Γ is the *type environment*: a map from $\langle \mathbf{id} \rangle$ to τ (type)

E.g., [$\mathbf{x} \leftarrow \mathbf{boolean}, \mathbf{y} \leftarrow \mathbf{number}$]

Similar in spirit to a deferred substitution, but maps to types, not values

Inference rules

Judgments are typically defined as a set of inference rules

$$\frac{A \quad B}{C}$$

This is a *rule*, which says: If I know A and B, then I can conclude C

- There could be 0 or more things above the bar

Terminology: A and B are *premises* C is the *conclusion*

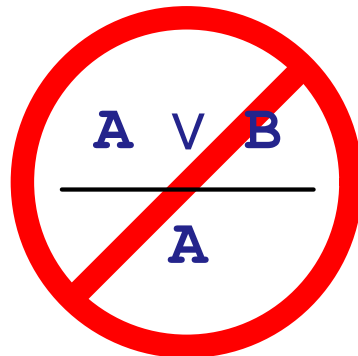
Inference Rule Example: Logical Or

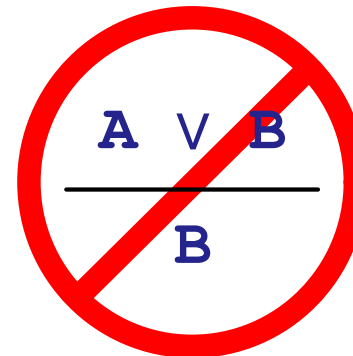
These two rules *together* describe logical or

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

These two are false inferences


$$\frac{A \vee B}{A}$$


$$\frac{A \vee B}{B}$$

Type Rules

$\Gamma \vdash \text{<num>} : \textit{number}$

$\Gamma \vdash \text{true} : \textit{boolean}$

$\Gamma \vdash \text{false} : \textit{boolean}$

$\Gamma \vdash \mathbf{e}_1 : \textit{number} \quad \Gamma \vdash \mathbf{e}_2 : \textit{number}$

$\Gamma \vdash \{+ \mathbf{e}_1 \mathbf{e}_2\} : \textit{number}$

(These have no premises)

$1 : \textit{number}$

$\text{true} : \textit{boolean}$

$1 : \textit{number} \quad 2 : \textit{number}$

$\{+ 1 2\} : \textit{number}$

$1 : \textit{number} \quad \text{false} : \textit{boolean}$

$\{+ 1 \text{false}\} : \textbf{no type}$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{number}$

$\Gamma \vdash \text{true} : \text{boolean}$

$\Gamma \vdash \text{false} : \text{boolean}$

$\Gamma \vdash \mathbf{e}_1 : \text{number} \quad \Gamma \vdash \mathbf{e}_2 : \text{number}$

$\Gamma \vdash \{+ \mathbf{e}_1 \mathbf{e}_2\} : \text{number}$

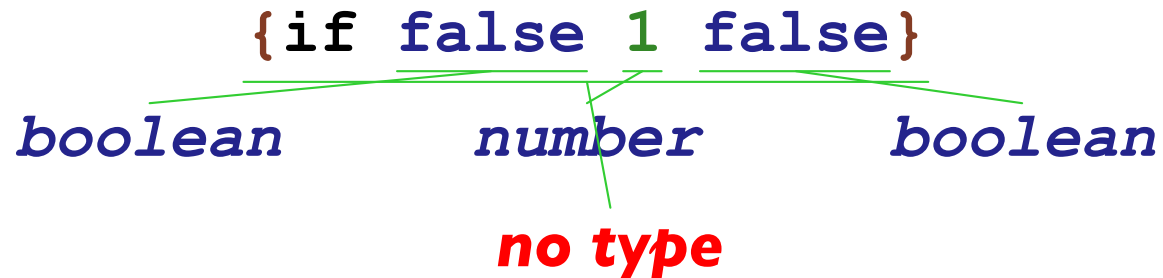
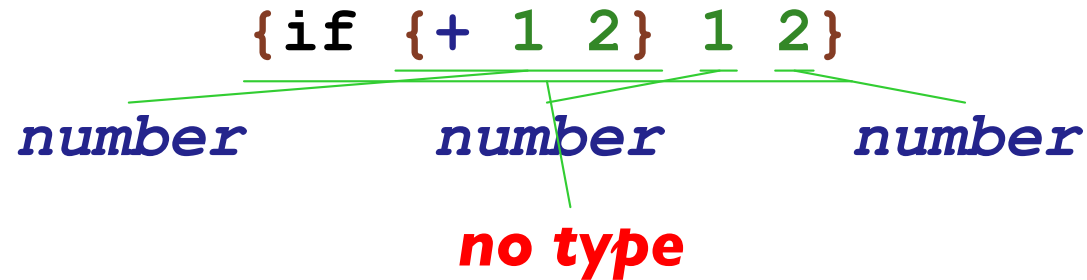
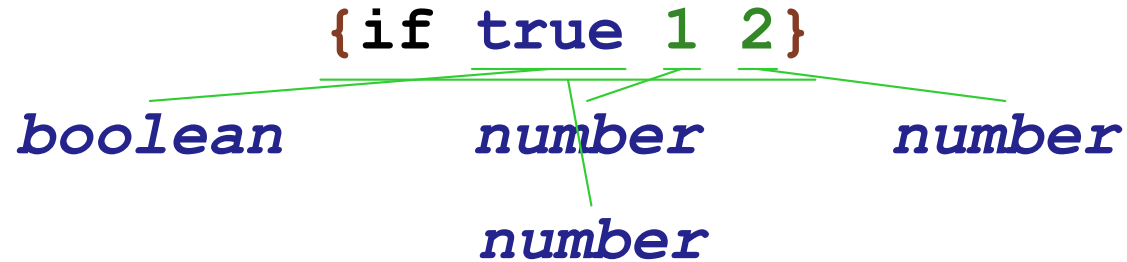
(These have no premises)

$1 : \text{number} \quad 2 : \text{number}$

$\{+ 1 2\} : \text{number} \quad 3 : \text{number}$

$\{+ \{+ 1 2\} 3\} : \text{number}$

Types: Conditionals



Conditional Type Rules

$$\Gamma \vdash \mathbf{e}_1 : \mathit{boolean} \qquad \Gamma \vdash \mathbf{e}_2 : \tau_0 \qquad \Gamma \vdash \mathbf{e}_3 : \tau_0$$

$$\Gamma \vdash \{\mathbf{if} \ \mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3\} : \tau_0$$

$$\mathit{true} : \mathit{boolean} \qquad 1 : \mathit{number} \qquad 2 : \mathit{number}$$

$$\{\mathbf{if} \ \mathit{true} \ 1 \ 2\} : \mathit{number}$$

$$\{+ \ 1 \ 2\} : \mathit{number} \qquad 1 : \mathit{number} \qquad 2 : \mathit{number}$$

$$\{\mathbf{if} \ \{+ \ 1 \ 2\} \ 1 \ 2\} : \mathbf{no \ type}$$

$$\mathit{false} : \mathit{boolean} \qquad 1 : \mathit{number} \qquad \mathit{false} : \mathit{boolean}$$

$$\{\mathbf{if} \ \mathit{false} \ 1 \ \mathit{false}\} : \mathbf{no \ type}$$

Types: Variables and Functions

x : no type

{fun {x : boolean} x}

boolean

(boolean → boolean)

{fun {x : boolean} {if x 1 2}}

boolean

number

number

number

(boolean → number)

Variable and Function Type Rules

$$[\dots \textcolor{blue}{\langle id \rangle} \leftarrow \tau \dots] \vdash \textcolor{blue}{\langle id \rangle} : \tau$$

$$\frac{\Gamma[\textcolor{blue}{\langle id \rangle} \leftarrow \tau_1] \vdash \textcolor{blue}{e} : \tau_2}{\Gamma \vdash \{ \textcolor{brown}{fun} \{ \textcolor{blue}{\langle id \rangle} : \tau_1 \} \textcolor{brown}{e} \} : (\tau_1 \rightarrow \tau_2)}$$

$$\emptyset \vdash \textcolor{blue}{x} : \textcolor{red}{no type}$$

$$\frac{[\textcolor{blue}{x} \leftarrow \textcolor{blue}{boolean}] \vdash \textcolor{blue}{x} : \textcolor{blue}{boolean}}{\emptyset \vdash \{ \textcolor{brown}{fun} \{ \textcolor{blue}{x} : \textcolor{blue}{boolean} \} \textcolor{brown}{x} \} : (\textcolor{blue}{boolean} \rightarrow \textcolor{blue}{boolean})}$$

$$\frac{\frac{[\textcolor{blue}{x} \leftarrow \textcolor{blue}{boolean}] \vdash \textcolor{blue}{x} : \textcolor{blue}{boolean} \quad [\textcolor{blue}{x} \leftarrow \textcolor{blue}{boolean}] \vdash \textcolor{green}{1} : \textcolor{blue}{number} \quad [\textcolor{blue}{x} \leftarrow \textcolor{blue}{boolean}] \vdash \textcolor{green}{2} : \textcolor{blue}{number}}{[\textcolor{blue}{x} \leftarrow \textcolor{blue}{boolean}] \vdash \{ \textcolor{brown}{if} \textcolor{blue}{x} \textcolor{green}{1} \textcolor{green}{2} \} : \textcolor{blue}{number}}}{\emptyset \vdash \{ \textcolor{brown}{fun} \{ \textcolor{blue}{x} : \textcolor{blue}{boolean} \} \{ \textcolor{brown}{if} \textcolor{blue}{x} \textcolor{green}{1} \textcolor{green}{2} \} \} : (\textcolor{blue}{boolean} \rightarrow \textcolor{blue}{number})}$$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{boolean} \rightarrow \text{number}) \quad \text{boolean}}$
number

$\frac{\{\{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{boolean} \rightarrow \text{number}) \quad \text{number}}$
no type

$\frac{\{7 \ 5\}}{\text{number} \quad \text{number}}$
no type

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash \text{true} : \text{boolean}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{number}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash 5 : \text{number}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \ \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{number} \quad \emptyset \vdash 5 : \text{number}}{\emptyset \vdash \{7 \ 5\} : \text{no type}}$$

Types: Multiple Arguments

`{fun {x : number y : number} {+ x y}}`

number *number*
number

$(\text{number number} \rightarrow \text{number})$

`{{fun {x : number y : number} {+ x y}} 5 6}`

$(\text{number number} \rightarrow \text{number})$ *number* *number*
number

`{{fun {x : number y : number} {+ x y}} 5}`

$(\text{number number} \rightarrow \text{number})$ *number*

no type

Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_1 \leftarrow \tau_1 \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle_1 : \tau_1 \dots \langle \text{id} \rangle_n : \tau_n \} \mathbf{e} \} : (\tau_1 \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \mathbf{e}_1 : \tau_1 \quad \dots \quad \Gamma \vdash \mathbf{e}_n : \tau_n}{\Gamma \vdash \{ \mathbf{e}_0 \ \mathbf{e}_1 \ \dots \ \mathbf{e}_n \} : \tau_0}$$