Transformations I

Lecture Set 5

CS5600***Computer Graphics***

by

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Transformations and Matrices

- Transformations are functions
- Matrices are functions representations
- Matrices represent linear transf's
- 2 x 2 Matrices == 2D Linear Transf's

What is a 2D Linear Transf ?

Recall from Linear Algebra:

Definition: $T(ax + y) = aT(x) + T(y)$
for scalar a and vectors x and y

Example: Scale in x

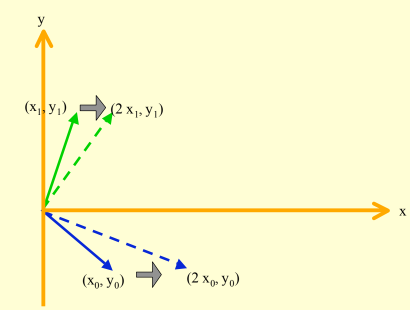
Scale in x, by 2:

$(2\ (x_o + x_1),\ y_0 + y_1) = (2x_o,\ y_o) + (2x_1,\ y_1)$

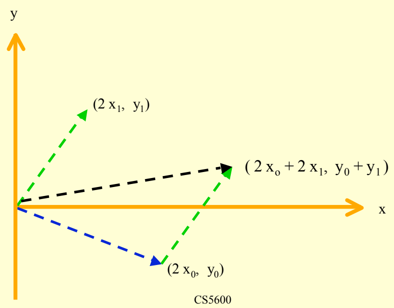
Example: Scale in x by 2

What is the graphical view?

Scale in x by 2



$(2x_o + 2x_1,\ y_0 + y_1)$



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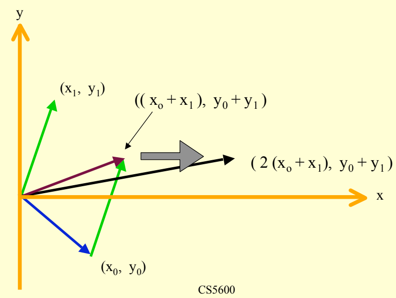
$(2 (x_o + x_1),\ y_0 + y_1)$



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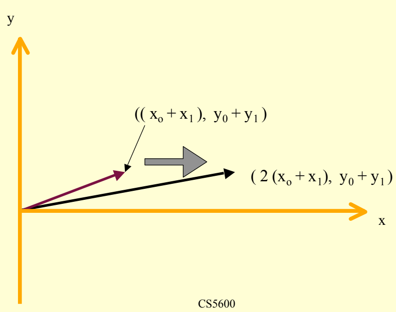
$(2 (x_o + x_1),\ y_0 + y_1)$



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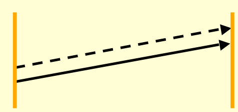
Summary on Scale

- “Scale then add,” is same as
- “Add then scale”



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Matrix Representation

Scale in x by 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

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Matrix Representation

Scale in y by 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

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Matrix Representation

Overall Scale by 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

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Matrix Representation Showing Same

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix} = \begin{bmatrix} 2(x_0 + x_1) \\ y_0 + y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_0 + 2x_1 \\ y_0 + y_1 \end{bmatrix}$$

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What about Rotation?

Is it linear?

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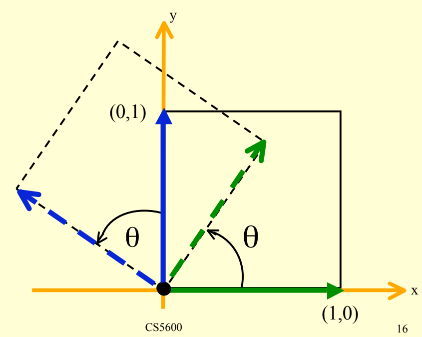
Rotate by θ



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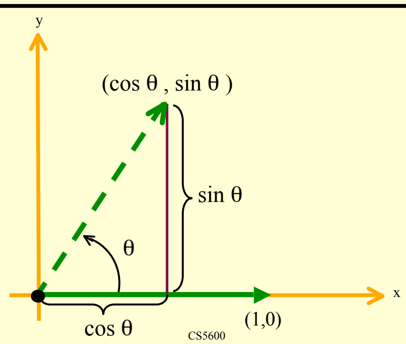
Rotate by θ: 1st Quadrant



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Rotate by θ: 1st Quadrant

(1, 0) → (cos θ , sin θ)

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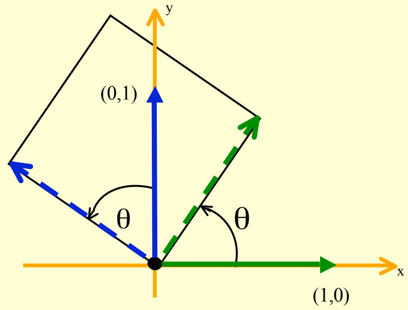
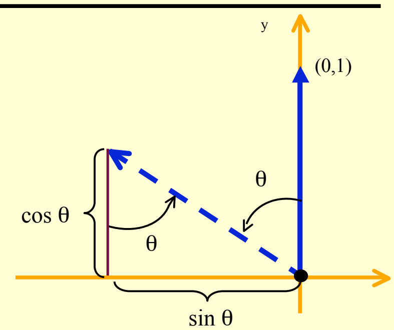
Rotate by θ: 2nd Quadrant



Rotate by θ: 2nd Quadrant



Rotate by θ: 2nd Quadrant

(0, 1) → (-sin θ , cos θ)

Summary of Rotation by θ

(1, 0) → (cos θ , sin θ)

(0, 1) → (-sin θ , cos θ)

Summary (Column Form)

$$\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Using Matrix Notation

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

(Note that unit vectors simply copy columns)

General Rotation by θ Matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin + y\cos\theta \end{bmatrix}$$

Who had linear algebra?

Who understand matrices?

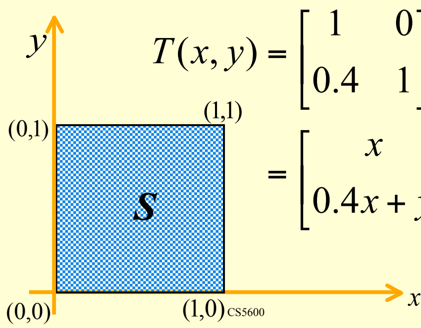
What do the off diagonal elements do?

Off Diagonal Elements

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

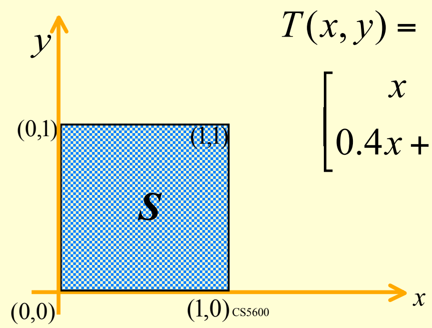
$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ bx + y \end{bmatrix}$$

Example 1



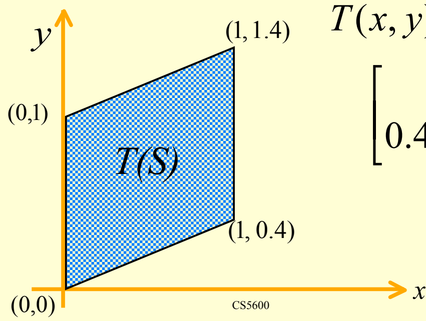
$$T(x,y) = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0.4x + y \end{bmatrix}$$

Example 1



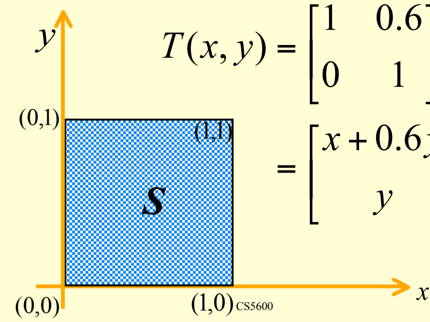
$$T(x,y) = \begin{bmatrix} x \\ 0.4x + y \end{bmatrix}$$

Example 1



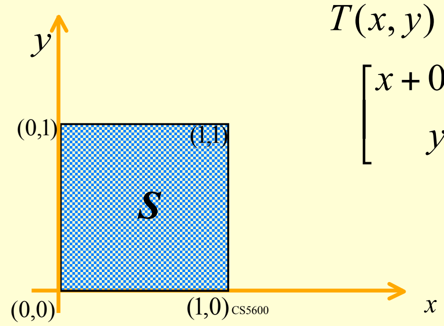
$$T(x,y) = \begin{bmatrix} x \\ 0.4x + y \end{bmatrix}$$

Example 2



$$T(x,y) = \begin{bmatrix} 1 & 0.6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 0.6y \\ y \end{bmatrix}$$

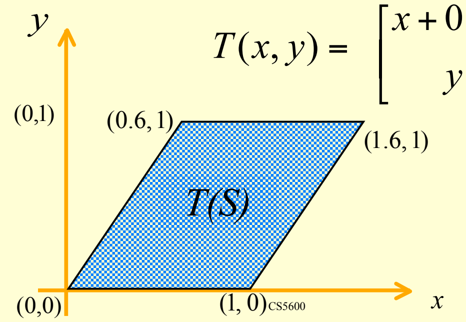
Example 2



$$T(x,y) = \begin{bmatrix} x + 0.6y \\ y \end{bmatrix}$$

Example 2

$$T(x,y) = \begin{bmatrix} x + 0.6y \\ y \end{bmatrix}$$



Summary

Shear in *x:*

$$Sh_x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

Shear in *y:*

$$Sh_y = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ bx + y \end{bmatrix}$$

Double Shear

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} (1+\mathrm{ab}) & a \\ b & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & (1+\mathrm{ab}) \end{bmatrix}$$

Sample Points: unit inverses

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}\begin{bmatrix}-a\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\b & 1\end{bmatrix}\begin{bmatrix}1\\-b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

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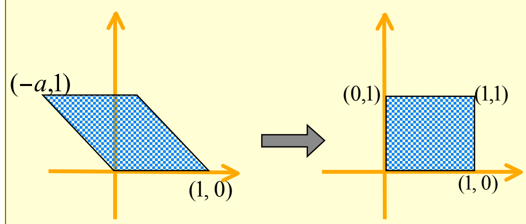
Geometric View of Shear in *x*



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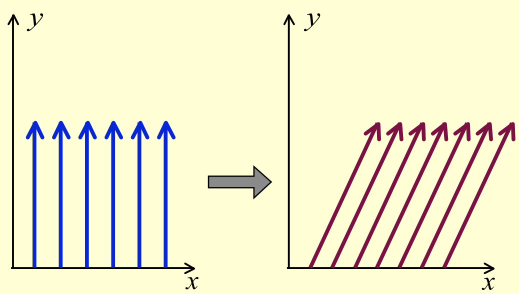
Another Geometric View of Shear in *x*



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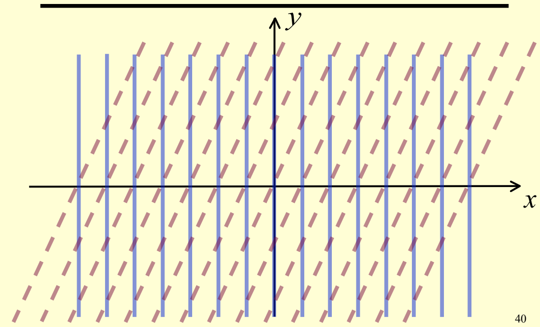
Another Geometric View of Shear in *x*



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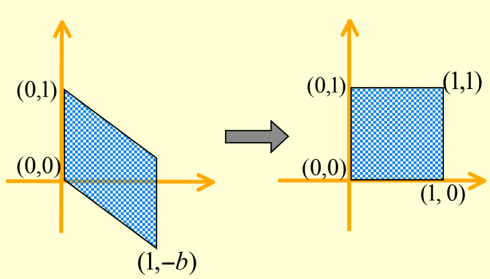
Geometric View of Shear in *y*



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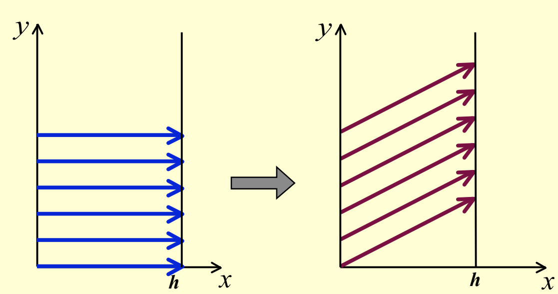
Another Geometric View of Shear in *y*



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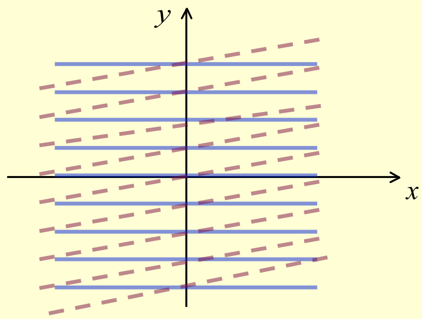
Another Geometric View of Shear in y



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"Lazy 1"

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Translation in x

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y \\ 1 \end{bmatrix}$$

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Translation in x

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y + d_y \\ 1 \end{bmatrix}$$

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Homogeneous Coordinates

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Homogeneous Coordinates

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}, \text{ for } \lambda \neq 0$$

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Homogeneous Coordinates

For $\lambda \neq 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \frac{1}{\lambda} \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

Homogeneous term effects overall scaling

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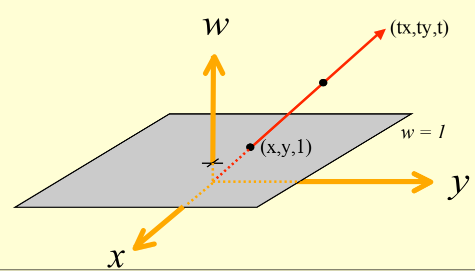
Homogeneous Coordinates

An infinite number of points correspond to (x,y,1).

They constitute the whole line (tx,ty,t).



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We've got Affine Transformations

Linear + Translation

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Next Class: Compound Transformations

- Build up compound transformations by concatenating elementary ones
- Use for complicated motion
- Use for complicated modeling

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