Early Developments of Logical Relations

Logical Relations for the Simply-Typed Lambda Calculus and System F

Shu-Hung You

1 Strong Normalization via Logical Predicates

```
@book{Pierce:2002:TPL:509043,
 author = {Pierce, Benjamin C.},
 title = {Types and Programming Languages},
 year = {2002},
 isbn = {0262162091, 9780262162098},
 edition = {1st},
 publisher = {The MIT Press},
}
```

In *Chapter 12: Normalization*, Pierce presents a proof of strong normalization for the simply-typed lambda calculus (STLC). The presentation shows the common structure of proofs involving logical relations: (i) define a family of type-indexed relations on closed terms (ii) show that the relation implies the desired conclusion (iii) show that well-typed terms are in the relation.

The first proof of strong normalization for STLC was given by W. W. Tait in the paper *Intensional Interpretations of Functionals of Finite Type I* in 1967. The proof technique Tait used evolves to modern-day logical relations.

2 Logical Relations in Denotational Semantics

```
@book{book,
  author = {Streicher, Thomas},
  year = {2006},
  month = {12},
  title = {Domain-theoretic foundations of functional programming},
  isbn = {978-981-270-142-8},
  doi = {10.1142/6284}
}
```

In *Chapter 4. Computational Adequacy*, Streicher gives a proof of adequacy for the Scott-domain based denotational semantics of Plotkin's PCF. The proof involves defining a logical relation between programs and denotations. In *Chapter 7. Logical Relations*, Streicher discusses how logical relations can be used to characterize definability of elements in Scott domains.

Plotkin was the first to use logical relations to characterize definability in *Lambda-definability and logical relations* in 1973. The idea was further developed by Statman in *Logical Relations and the Typed Lambda-Calculus* in 1985.

Moreover, Plotkin presented PCF and was the first to prove the adequacy of its denotational semantics using logical predicates in the 1977 paper *LCF considered as a programming language.*

3 Proving Type Soundness through Logical Predicates

```
@phdthesis{Ahmed:2004:STM:1037736,
 author = {Ahmed, Amal Jamil},
 title = {Semantics of Types for Mutable State},
 year = {2004},
 note = {AAI3136691},
 publisher = {Princeton University},
 address = {Princeton, NJ, USA},
}
```

The first half of *Chapter 2. Foundational Proofs of Safety* in the dissertation includes a brief introduction to the application of logical relations to proving type safety of STLC. The dissertation explores the semantic models of types using Kripke and step-indexed logical relations for languages with mutation. These are beyond the scope of our discussion.

4 Contextual Equivalence from Logical Relations

```
@inproceedings{Pitts:1998:ETL:646252.686023,
 author = {Pitts, Andrew M.},
 title = {Existential Types: Logical Relations and Operational Equivalence},
 series = {ICALP ’98},
 year = {1998},
 isbn = {3-540-64781-3},
 pages = {309--326},
 publisher = {Springer-Verlag},
}
```

```
@book{Pierce:2004:ATT:1076265,
 author = {Pierce, Benjamin C.},
 title = {Advanced Topics in Types and Programming Languages},
 year = {2004},
 isbn = {0262162288},
 publisher = {The MIT Press},
}
```

(**Summary**) In this work, Pitts develops a relational interpretation of types for the extended System F. Pitts demonstrates that the proposed relation is a subset of contextual equivalence and proves the fundamental lemma that well-typed terms are related to themselves. This induces a powerful equality principle for existential types.

An expanded and revised version of the paper appears as *Chapter 7, Typed Operational Reasoning* in the book *Advanced Topics in Types and Programming Languages.*

(**Evaluation**) The relational interpretation of types dates back to Reynolds's 1983 paper, *Types, Abstraction and Parametric Polymorphism.* However, Reynolds focuses more on abstraction and representation independence. In Pitts's paper, the logical relation is developed in an operational, syntactical setting and the connection to contextual equivalence is made explicit. This is one of the early papers that employed logical relations as a powerful tool to reason about contextual equivalence.