

Hypothesis Testing and Computational Learning Theory

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Overview

- Hypothesis Testing: How do we know our learners are “good” ?
 - What does performance on test data imply/guarantee about future performance?
- Computational Learning Theory: Are there general laws that govern learning?
 - ***Sample Complexity***: How many training examples are needed to learn a successful hypothesis?
 - ***Computational Complexity***: How much computational effort is needed to learn a successful hypothesis?

Some terms

X is the set of all possible instances

C is the set of all possible concepts c
where $c : X \rightarrow \{0,1\}$

H is the set of hypotheses considered
by a learner, $H \subseteq C$

L is the learner

D is a probability distribution over X
that generates observed instances

Definition

- The **true error** of hypothesis h , with respect to the target concept c and observation distribution D is the probability that h will misclassify an instance drawn according to D

$$error_D \equiv P_{x \in D} [c(x) \neq h(x)]$$

- In a perfect world, we'd like the true error to be 0

Definition

- The **sample error** of hypothesis h , with respect to the target concept c and sample S is the proportion of S that that h misclassifies:

$$error_S(h) = 1/|S| \sum_{x \in S} \delta(c(x), h(x))$$

where $\delta(c(x), h(x))$ returns 1 iff $c(x) \neq h(x)$

Problems Estimating Error

1. *Bias*: If S is training set, $error_S(h)$ is optimistically biased

$$bias \equiv E[error_S(h)] - error_{\mathcal{D}}(h)$$

For unbiased estimate, h and S must be chosen independently

2. *Variance*: Even with unbiased S , $error_S(h)$ may still *vary* from $error_{\mathcal{D}}(h)$

Example on Independent Test Set

Hypothesis h misclassifies 12 of the 40 examples in S

$$error_S(h) = \frac{12}{40} = .30$$

What is $error_{\mathcal{D}}(h)$?

Estimators

Experiment:

1. choose sample S of size n according to distribution $\mathcal{D}$
2. measure $error_S(h)$

$error_S(h)$ is a random variable (i.e., result of an experiment)

$error_S(h)$ is an unbiased *estimator* for $error_{\mathcal{D}}(h)$

Given observed $error_S(h)$ what can we conclude about $error_{\mathcal{D}}(h)$?

Confidence Intervals

If

- S contains n examples, drawn independently of h and each other
- $n \geq 30$ and $n \cdot \text{error}_S(h)$, $n \cdot (1 - \text{error}_S(h))$ each > 5

Then

- With approximately 95% probability, $\text{error}_D(h)$ lies in interval

$$\text{error}_S(h) \pm 1.96 \sqrt{\frac{\text{error}_S(h)(1 - \text{error}_S(h))}{n}}$$

Confidence Intervals

- Under same conditions...
- With approximately $N\%$ probability, $error_{\mathcal{D}}(h)$ lies in interval

$$error_S(h) \pm z_N \sqrt{\frac{error_S(h)(1 - error_S(h))}{n}}$$

where

$N\%:$	50%	68%	80%	90%	95%	98%	99%
$z_N:$	0.67	1.00	1.28	1.64	1.96	2.33	2.58

Life Skills

- “Convincing demonstration” that certain enhancements improve performance?
- Use online Fisher Exact or Chi Square tests to evaluate hypotheses, e.g:
 - <http://people.ku.edu/~preacher/chisq/chisq.htm>

Overview

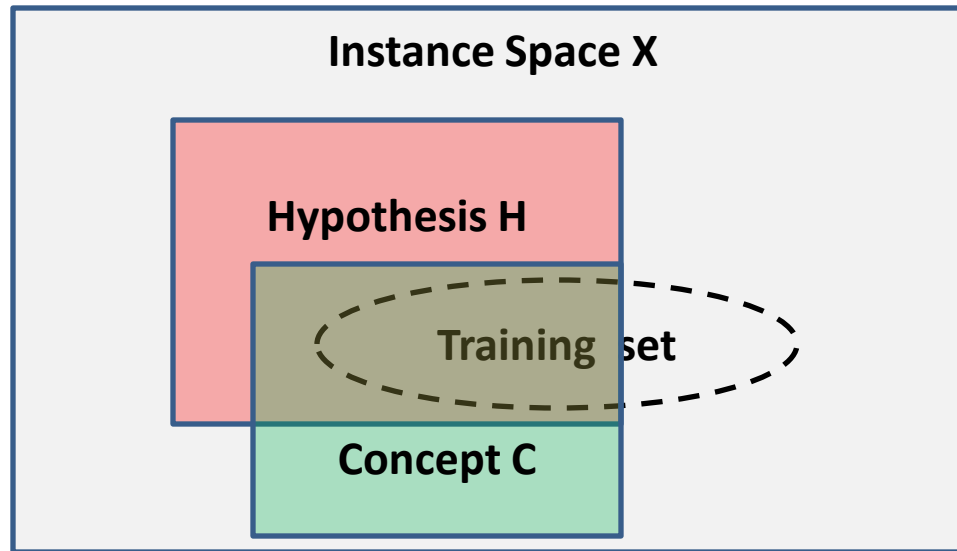
- Hypothesis Testing: How do we know our learners are “good” ?
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Computational Learning Theory

- Are there general laws that govern learning?
 - ***No Free Lunch Theorem:*** The expected accuracy of *any* learning algorithm across all concepts is 50%.
- But can we still say something positive?
 - Yes.
 - *Probably Approximately Correct* (PAC) learning

The world isn't perfect

- If we can't provide every instance for training, a consistent hypothesis may have error on unobserved instances.



- How many training examples do we need to bound the likelihood of error to a reasonable level?
When is our hypothesis Probably Approximately Correct (PAC)?

Definitions

- A hypothesis is **consistent** if it has zero error on training examples
- The **version space** ($VS_{H,T}$) is the set of all hypotheses consistent on training set T in our **hypothesis space** H
 - (reminder: hypothesis space is the set of concepts we're considering, e.g. depth-2 decision trees)

Definition: ε -exhausted

IN ENGLISH:

The set of hypotheses consistent with the training data T is ε -exhausted if, when you test them on the actual distribution of instances, all consistent hypotheses have error below ε

IN MATH:

$VS_{H,T}$ is ε - exhausted for concept c

and sample distribution on D , if....

$$\forall h \in VS_{H,T}, error_D(h) < \varepsilon$$

A Theorem

If hypothesis space H is finite, & training set T contains m independent randomly drawn examples of concept c

THEN, for any $0 \leq \varepsilon \leq 1$...

$$P(VS_{H,T} \text{ is NOT } \varepsilon\text{-exhausted}) \leq |H|e^{-\varepsilon m}$$

Proof of Theorem

If hypothesis h has true error ε , the probability of it getting a single random example right is :

$$P(h \text{ got 1 example right}) = 1 - \varepsilon$$

Ergo the probability of h getting m examples right is :

$$P(h \text{ got } m \text{ examples right}) = (1 - \varepsilon)^m$$

Proof of Theorem

If there are k hypotheses in H with error at least ε , call the probability at least of those k hypotheses got m instances right $P(\text{at least one bad } h \text{ looks good})$.

This prob. is BOUNDED by $k(1-\varepsilon)^m$

$$P(\text{at least one bad } h \text{ looks good}) \leq k(1-\varepsilon)^m$$



"Union" bound

Proof of Theorem (continued)

Since $k \leq |H|$, it follows that $k(1-\varepsilon)^m \leq |H|(1-\varepsilon)^m$

If $0 \leq \varepsilon \leq 1$, then $(1 - \varepsilon) \leq e^{-\varepsilon}$

Therefore. ..

$$P(\text{at least one bad } h \text{ looks good}) \leq k(1-\varepsilon)^m \leq |H|(1-\varepsilon)^m \leq |H|e^{-\varepsilon m}$$

Proof complete!

We now have a bound on the likelihood that a hypothesis is consistent with the training data will have error $\geq \varepsilon$

Using the theorem

Let's rearrange to see

$$|H|e^{-\varepsilon m} \leq \delta$$

how many training

$$\ln(|H|e^{-\varepsilon m}) \leq \ln(\delta)$$

examples we need

$$\ln(|H|) + \ln(e^{-\varepsilon m}) \leq \ln(\delta)$$

to set a bound δ on

$$\ln(|H|) - \varepsilon m \leq \ln(\delta)$$

the likelihood our

$$\ln(|H|) - \ln(\delta) \leq \varepsilon m$$

true error is ε .

$$\frac{1}{\varepsilon} (\ln(|H|) - \ln(\delta)) \leq m$$

$$\frac{1}{\varepsilon} \left(\ln(|H|) + \ln\left(\frac{1}{\delta}\right) \right) \leq m$$

Probably Approximately Correct (PAC)

$$\frac{1}{\epsilon} \left(\ln(|H|) - \ln(\delta) \right) \leq m$$

The worst error
we'll tolerate

hypothesis
space size

The likelihood a
hypothesis consistent
with the training data
will have error ϵ

number of training examples

Using the bound

$$\frac{1}{\epsilon} \left(\ln(|H|) - \ln(\delta) \right) \leq m$$

Plug in ϵ , δ , and H to get a number of training examples m that will “guarantee” your learner will generate a hypothesis that is Probably Approximately Correct.

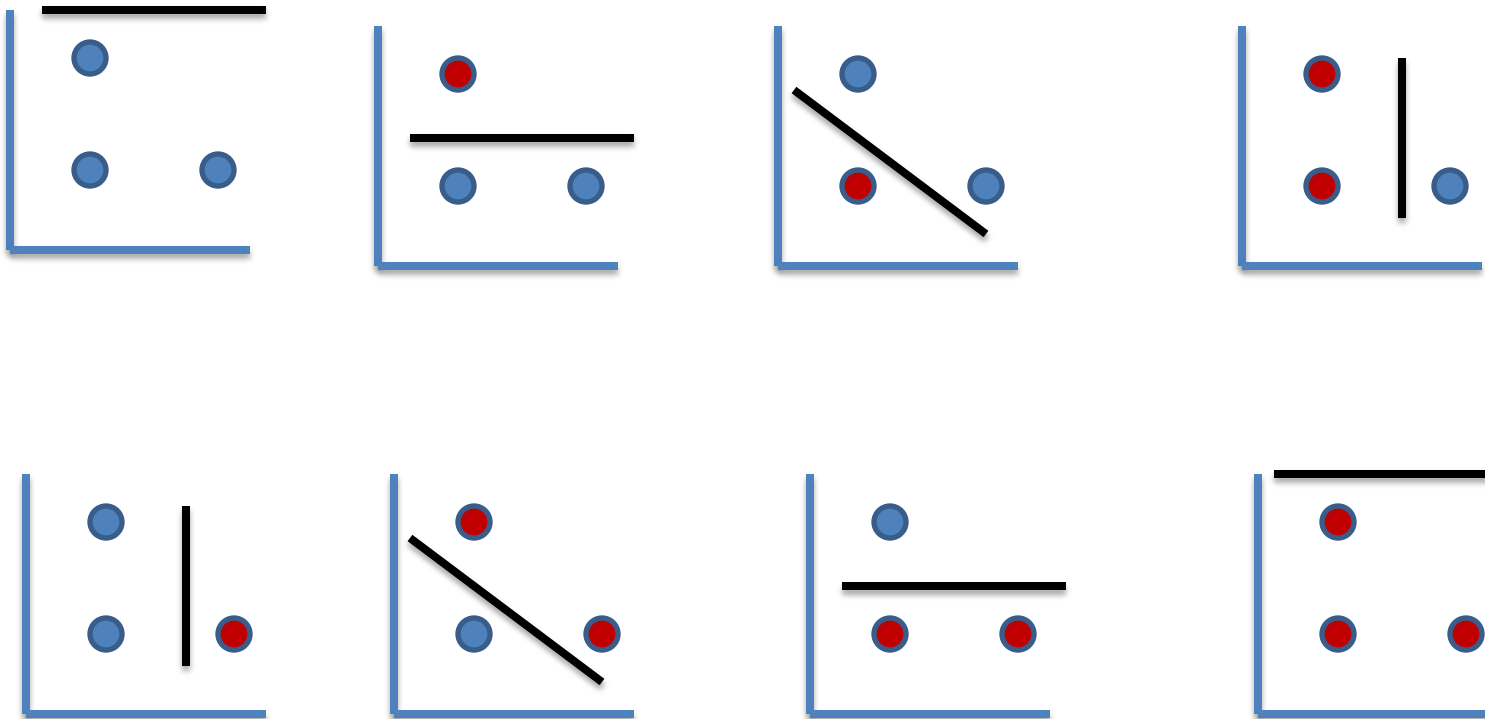
NOTE: This assumes that the concept is actually IN H , that H is finite, and that *your training set is drawn using distribution D*

Problems with PAC

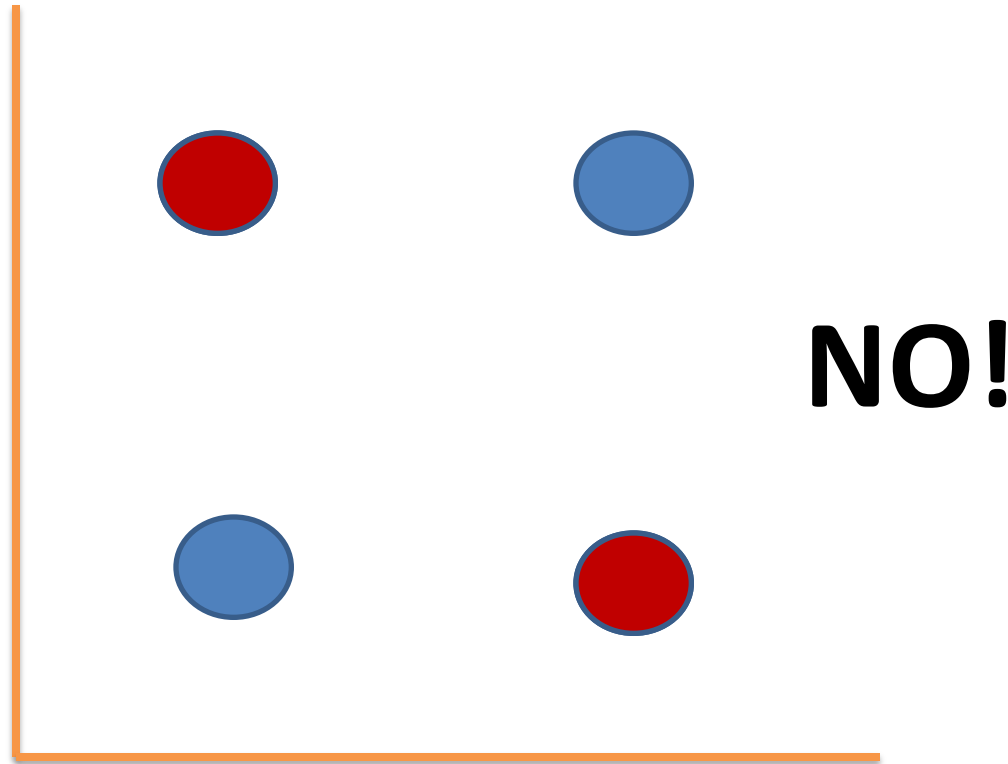
- The PAC Learning framework has 2 disadvantages:
 - 1) It can lead to weak bounds
 - 2) Sample Complexity bound cannot be established for infinite hypothesis spaces
- We introduce the VC dimension for dealing with these problems

Shattering

Def: A set of instances $\mathcal{S}$ is shattered by hypothesis set $\mathcal{H}$ iff for every possible concept c on $\mathcal{S}$ there exists a hypothesis h in $\mathcal{H}$ that is consistent with that concept.

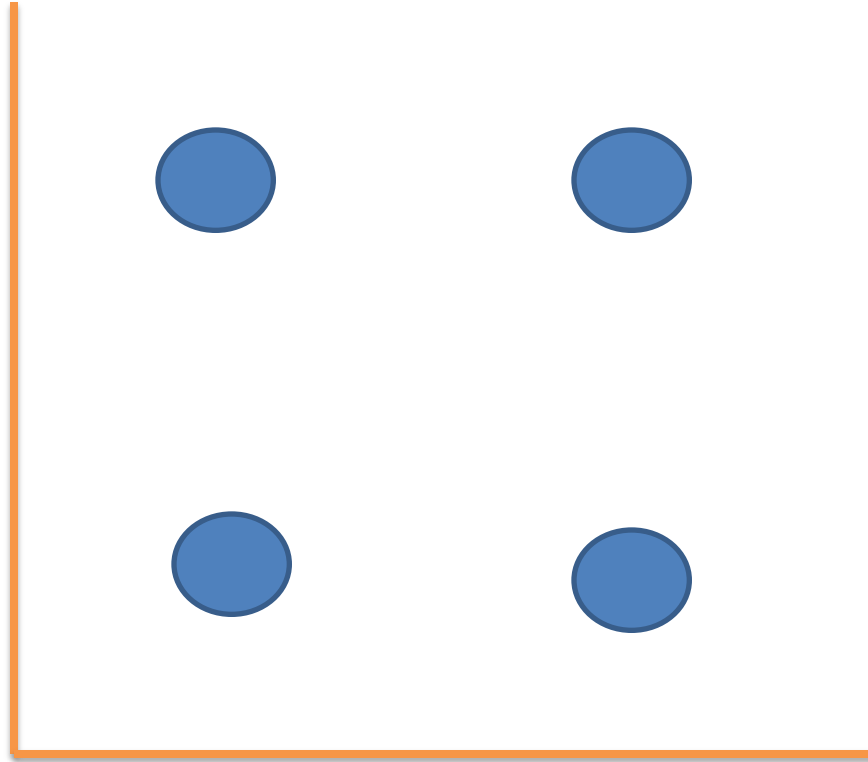


Can a linear separator shatter this?



The ability of H to shatter a set of instances is a measure of its capacity to represent target concepts defined over those instances

Can a quadratic separator shatter this?



This sounds like a homework problem....

Vapnik-Chervonenkis Dimension

Def: The **Vapnik-Chervonenkis dimension**, $VC(H)$ of hypothesis space H defined over instance space X is the size of the largest finite subset of X shattered by H . If arbitrarily large finite sets can be shattered by H , then $VC(H)$ is infinite.

How many training examples needed?

- Upper bound on m using $VC(H)$

This uses the
HYPOTHESIS
space

$$m \geq \frac{1}{\varepsilon} (4 \log_2 (2 / \delta) + 8 VC(H) \log_2 (13 / \varepsilon))$$

- Lower bound on m using $VC(C)$

This uses the
CONCEPT space

$$\max \left[\frac{1}{\varepsilon} \log(1 / \delta), \frac{VC(C) - 1}{32 \varepsilon} \right]$$

There's more on this in the textbook.