Outline

- Basics of probability
- Statistical estimation
- Bayes Nets
- Naïve Bayes
- Markov Nets (briefly, we will come back to this)
- **Inference**
- Learning
- Statistical Language Models

Inference: Variable Elimination

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Inference: Answering Queries

- Given:
  - A probability model
  - Subsets of random variables
    - ***Y*** (query) and
    - ***E*** (evidence) with assignments **e** to ***E***
- Find P(***Y*** | ***E*** = **e**)
- E.g.,
  - P(*Battery* | *Starts* = false)
  - P(*Disease* | *Symptoms* = **e**)
  - P(*StockMarketCrash* | *RecentPriceActivity* = **e**)

What else can we do with queries?

- Prioritizing info gathering
  - Which additional evidence would be most informative?
- Explanation
  - Why do I need a new fan belt?
- Sensitivity Analysis
  - Which variable values are most critical?

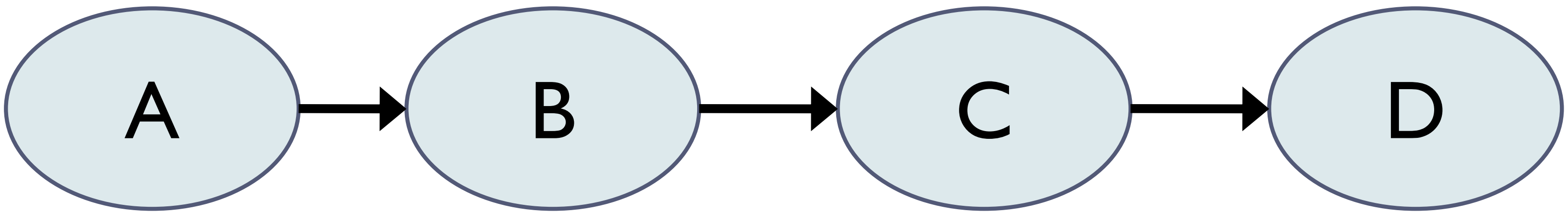
Gee, it's easy

- $P(\boldsymbol{Y} \mid \boldsymbol{E} = \mathbf{e}) = \dfrac{P(\boldsymbol{Y}, \mathbf{e})}{P(\mathbf{e})}$

- Given joint $P(\boldsymbol{y}, \mathbf{e}, \boldsymbol{w})$, we can compute r.h.s. by summing out $\boldsymbol{w}$, $\boldsymbol{y}$

But…

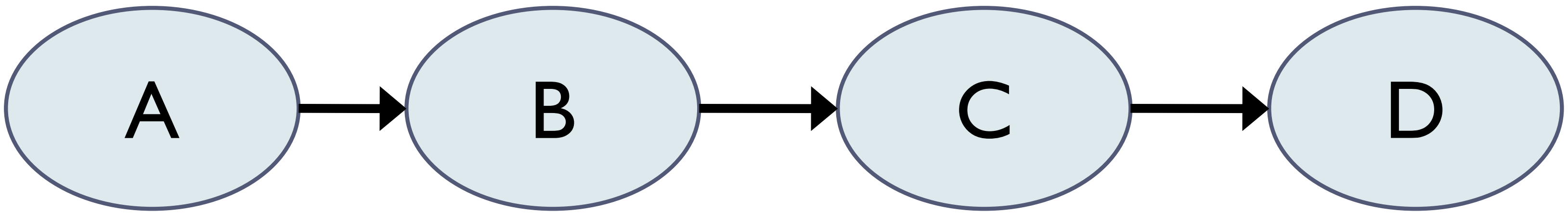
- Naïve summing is costly



  - $P(A, B, C, D) = P(A)\ P(B|A)\ P(C|B)\ P(D|C)$

- $P(D) = \Sigma_A\ \Sigma_B\ \Sigma_C\ P(A)\ P(B|A)\ P(C|B)\ P(D|C)$
  - $2^3 = 8$ combinations, $8*3 =$ **24** multiplications
  - **Exponential** in # of variables

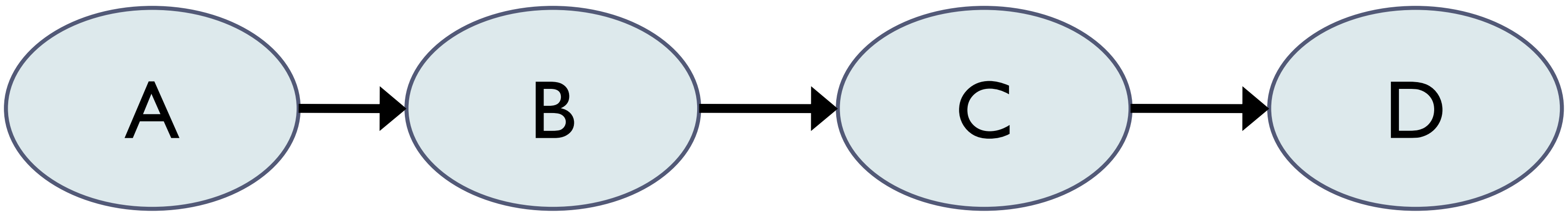
Variable Elimination



$$P(D) = \Sigma_A \Sigma_B \Sigma_C \, P(A) \, P(B|A) \, P(C|B) \, P(D|C)$$

$$= \Sigma_C \, P(D|C) \, \Sigma_B \, P(C|B) \, \underbrace{\Sigma_A \, P(B|A) \, P(A)}_{P(B)}$$

Variable Elimination



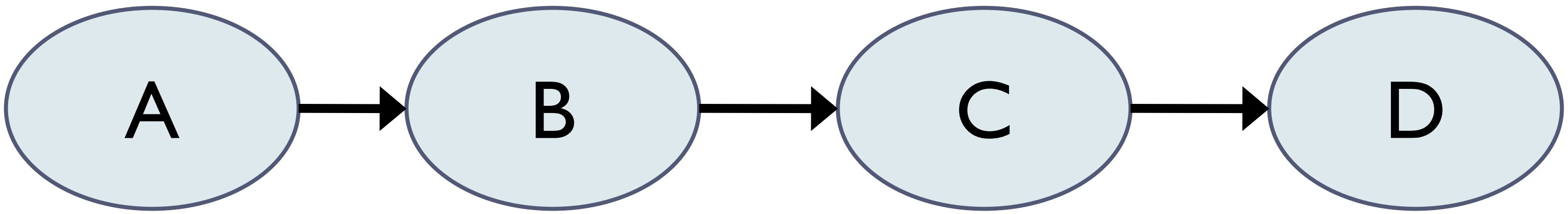
$$P(D) = \Sigma_A \Sigma_B \Sigma_C P(A)\, P(B|A)\, P(C|B)\, P(D|C)$$

$$= \Sigma_C P(D|C)\, \Sigma_B P(C|B)\, \Sigma_A P(B|A)\, P(A)$$

Has 2+2+2=**6** multiplications (vs. 24)

- For *n*-edge binary chain, only ***2n*** multiples

With evidence



$$P(D|A=a) = \Sigma_B \Sigma_C P(B|A=a)\, P(C|B)\, P(D|C)$$

$$= \Sigma_C P(D|C)\, \Sigma_B P(C|B)\, P(B|A=a)$$

Variable Elimination

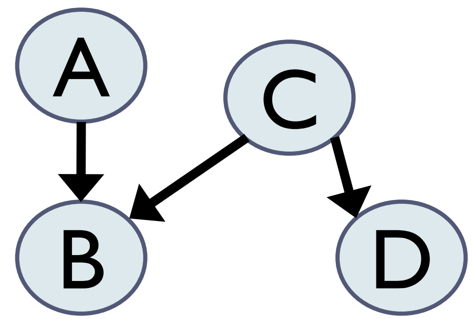
- Two steps:
  - Push summations as far as possible to right (assuming some ordering of variables)
  - Compute the sum

$$P(D|A=a) = \Sigma_B \, \Sigma_C \, P(D|C) \, P(C|B) \, P(B|A=a)$$

$$= \Sigma_C \, P(D|C) \, \Sigma_B \, P(C|B) \, P(B|A=a)$$

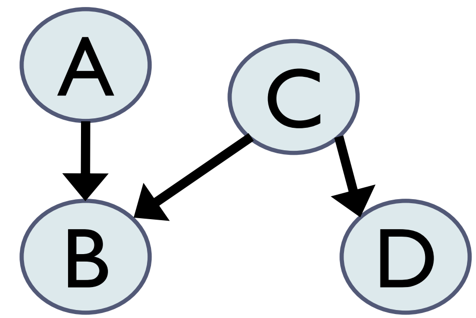
“Factors”

- $P(A, B, C, D)$
  $= \underbrace{P(A)}_{\phi_1} \cdot \underbrace{P(C)}_{\phi_2} \cdot \underbrace{P(B \mid A, C)}_{\phi_3} \cdot \underbrace{P(\mathrm{D} \mid \mathrm{C})}_{\phi_4}$



- Scope [ϕ_4] = $\{D, C\}$
- Variable Elimination: write out joint as factors
  - factor ϕ_i out of sum over X when $X \notin$ scope [ϕ_i]

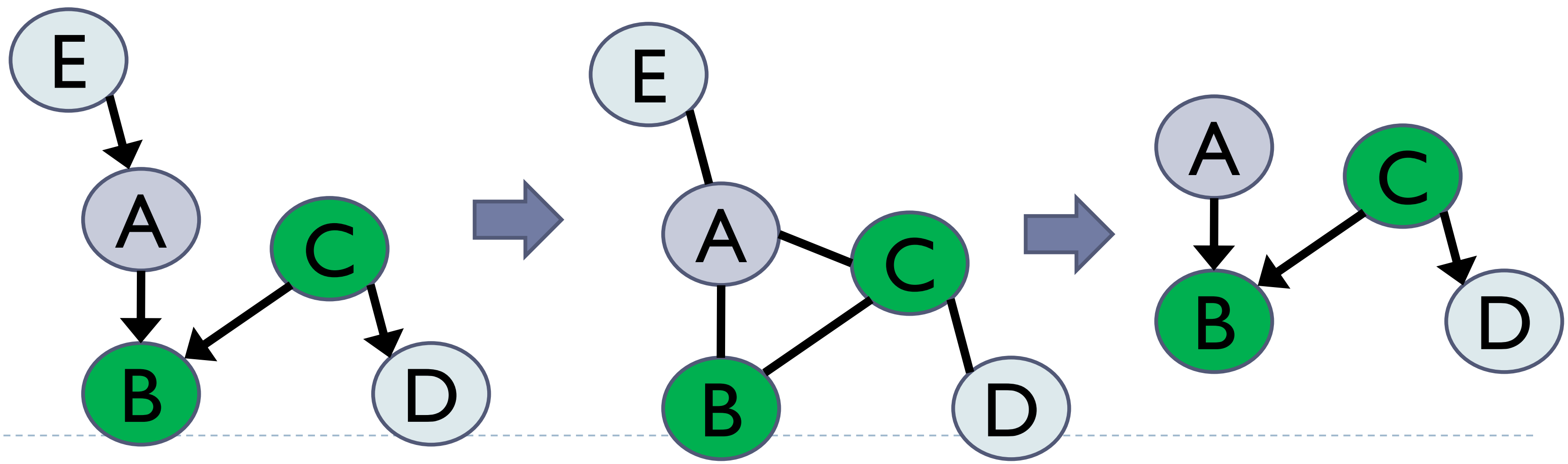
Discarding non-Ancestors



- $P(A, B, C, D)$
  $= P(A)\, P(C)\, P(B \mid A, C) P(D \mid C)$
- Query: $P(B, C \mid A{=}a)$
  $= \Sigma_D\, P(C)\, P(B \mid A{=}a, C) P(D \mid C)$
  $= P(C)\, P(B \mid A{=}a, C)\, \Sigma_D\, P(D \mid C)$
- $\Sigma_D\, P(D \mid C) = 1$ for all C, we can ignore it
- In general: when computing $P(\boldsymbol{Y} \mid \boldsymbol{E})$ we can ignore nodes not in $Ancestors(\boldsymbol{Y}, \boldsymbol{E})$

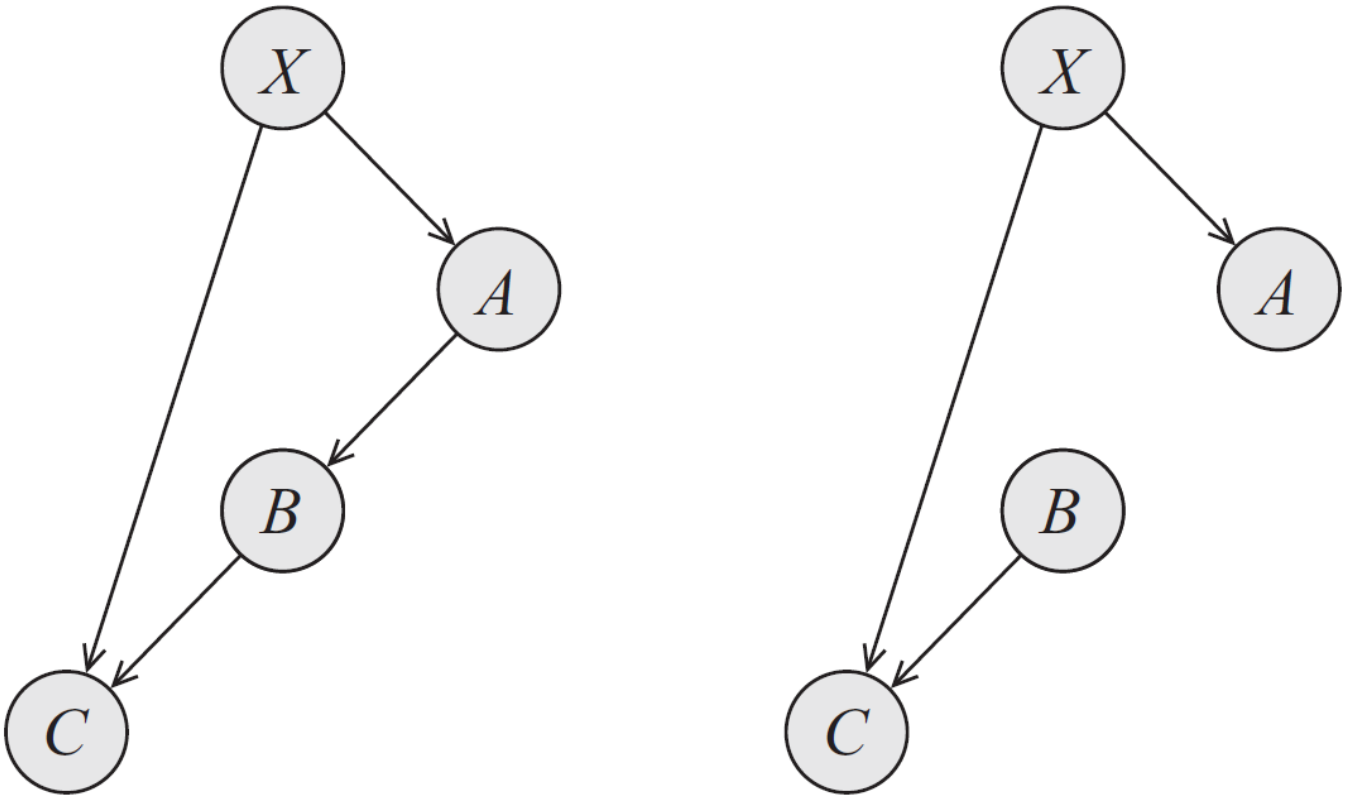
Discard by separation in Markov Network

- $P(A, B, C, D, E)$
  $= P(E)\ P(A|E)\ P(C)\ P(B \mid A, C) P(D \mid C)$
- Query: $P(B, C \mid A{=}a)$
  - Throw out variables separated from query by evidence in moral graph



Semantics of summed-out factors

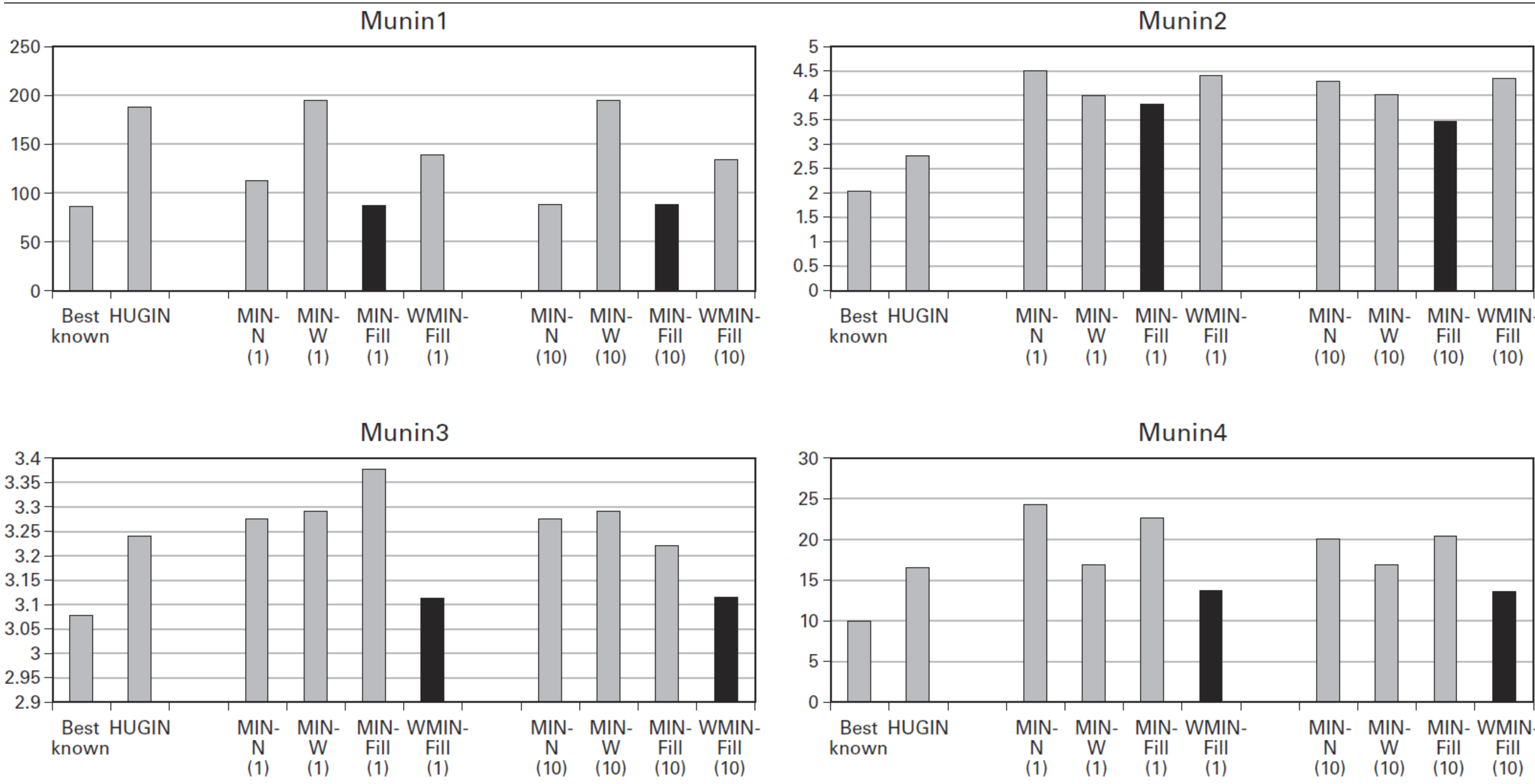
- Sums don't always correspond to simple conditional probabilities

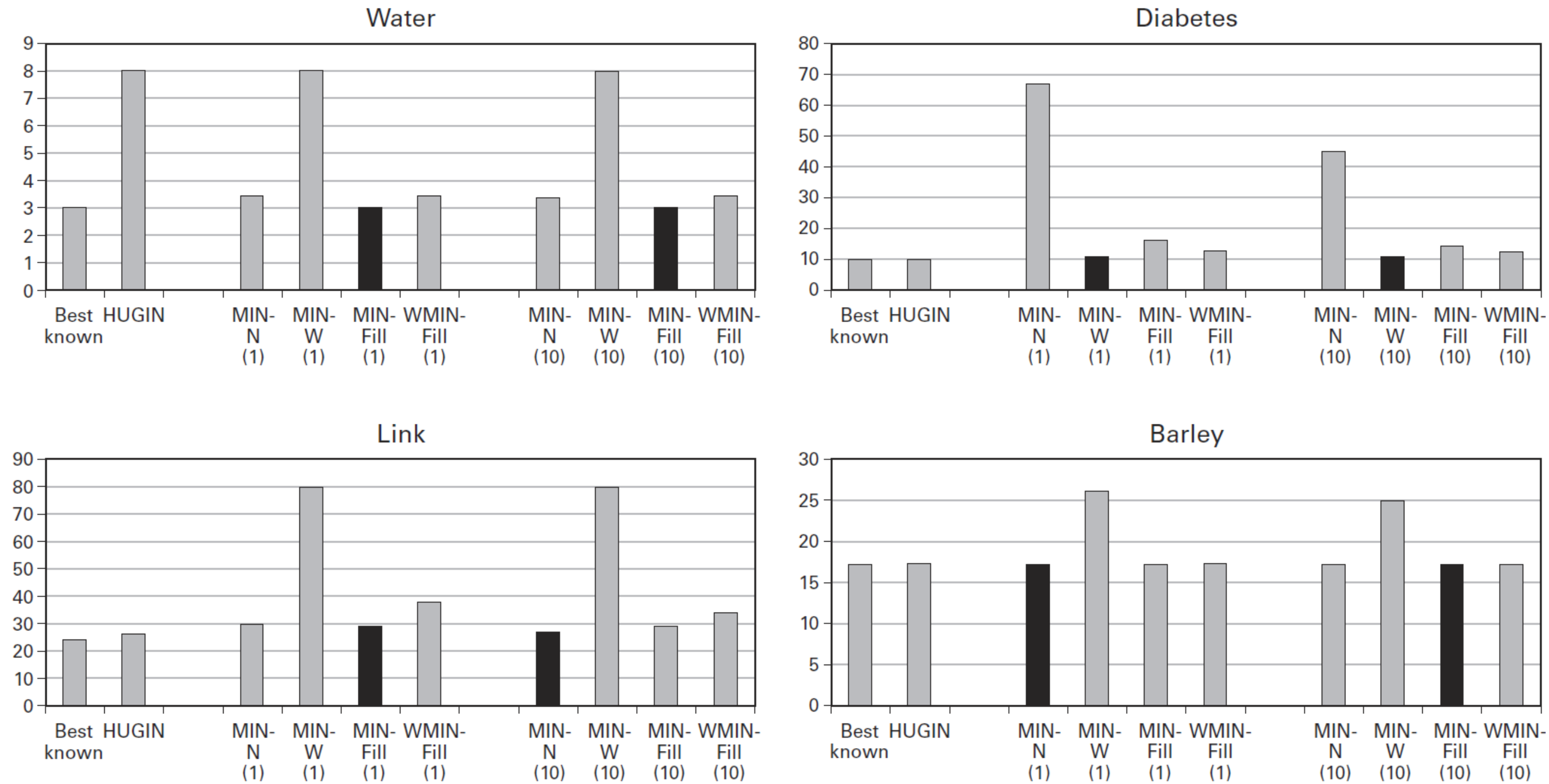


Complexity of Inference

- What does variable elimination buy us?
- It depends on the network
  - If the distribution doesn't factor well, elimination won't help
- Generally, Bayesian Inference is hard
- NP-complete problems can be reduced to it

- Ordering heuristics:
  - Min neighbors (weighted)
  - Min fill (weighted)


Munin1
250
200
150
100
50
0
Best known
HUGIN
MIN-N (1)
MIN-W (1)
MIN-Fill (1)
WMIN-Fill (1)
MIN-N (10)
MIN-W (10)
MIN-Fill (10)
WMIN-Fill (10)
Munin2
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Best known
HUGIN
MIN-N (1)
MIN-W (1)
MIN-Fill (1)
WMIN-Fill (1)
MIN-N (10)
MIN-W (10)
MIN-Fill (10)
WMIN-Fill (10)
Munin3
3.4
3.35
3.3
3.25
3.2
3.15
3.1
3.05
3
2.95
2.9
Best known
HUGIN
MIN-N (1)
MIN-W (1)
MIN-Fill (1)
WMIN-Fill (1)
MIN-N (10)
MIN-W (10)
MIN-Fill (10)
WMIN-Fill (10)
Munin4
30
25
20
15
10
5
0
Best known
HUGIN
MIN-N (1)
MIN-W (1)
MIN-Fill (1)
WMIN-Fill (1)
MIN-N (10)
MIN-W (10)
MIN-Fill (10)
WMIN-Fill (10)



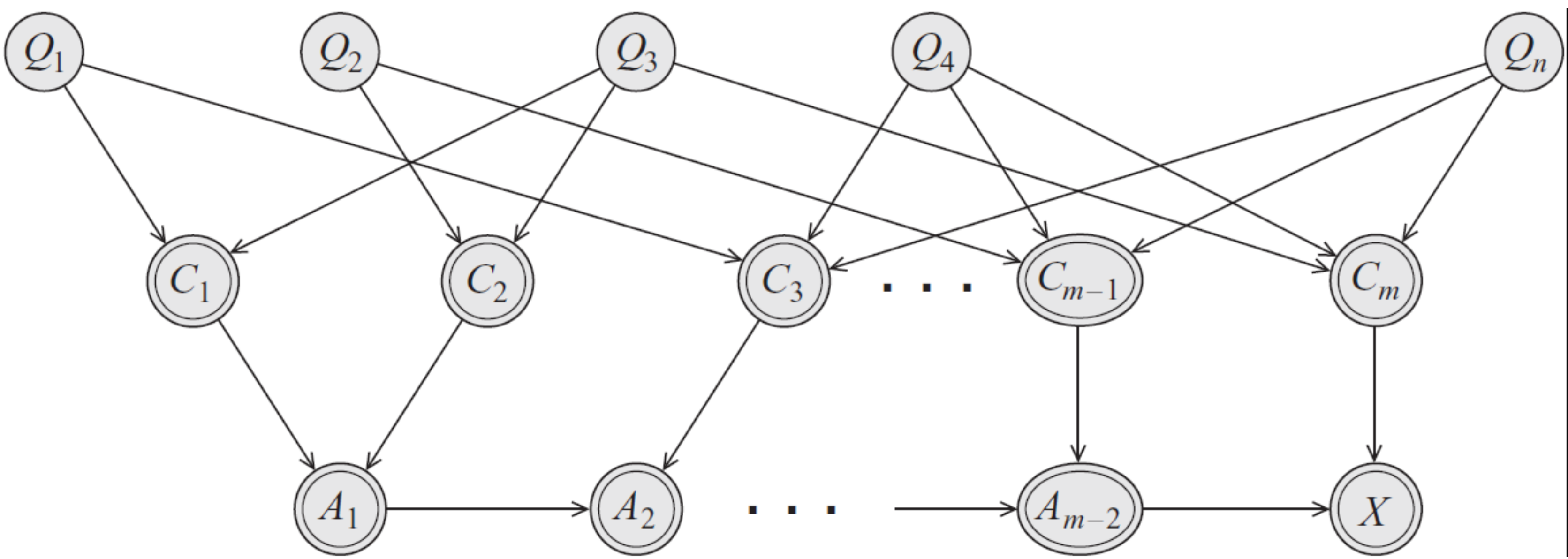
Water
Diabetes
Link
Barley
Best known
HUGIN
MIN-N (1)
MIN-W (1)
MIN-Fill (1)
WMIN-Fill (1)
MIN-N (10)
MIN-W (10)
MIN-Fill (10)
WMIN-Fill (10)

Reduction from Boolean Satisfiability (1)

- Boolean Satisfiability
  - Given a boolean formula in 3-CNF, e.g.:
    `(x1 v -x3 v x7) ^ (x4 v x5 v -x6) ^` …
    Is there an assignment to variables (i.e. xi = true|false) that makes the formula true?

Reduction from Boolean Satisfiability (2)

- **`(x1 v -x3 v x7) ^ (x4 v x5 v -x6)`**
  - Let Q_i **`= xi`**
  - C_i `=` clauses (e.g. **`(x1 v -x3 v x7)`**)
  - **`X =`** true *iff* all C_i are true, A_i's are "and" variables



Inference complexity details

- Actually #P-complete
  - Asking for probability ≈ counting number of satisfying assignments
- Even approximation is NP-hard
- (see book)