The Binary Heap

CS 214, Fall 2019
Implementing a priority queue

A (min-)priority queue provides these operations:

- **insert**: adds an element
- **remove_min**: removes the smallest element
Some implementation complexities

<table>
<thead>
<tr>
<th></th>
<th>insert</th>
<th>remove_min</th>
</tr>
</thead>
<tbody>
<tr>
<td>sorted list</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>unsorted list</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
</tr>
</tbody>
</table>
Some implementation complexities

<table>
<thead>
<tr>
<th>List Type</th>
<th>Insert Complexity</th>
<th>Remove Min Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sorted List</td>
<td>$\mathcal{O}(n)$</td>
<td>$\mathcal{O}(1)$</td>
</tr>
<tr>
<td>Unsorted List</td>
<td>$\mathcal{O}(1)$</td>
<td>$\mathcal{O}(n)$</td>
</tr>
<tr>
<td>Binary Heap</td>
<td>$\mathcal{O}(\log n)$</td>
<td>$\mathcal{O}(\log n)$</td>
</tr>
</tbody>
</table>


Introducing the binary heap

A binary heap is complete binary tree that is heap-ordered
A tree is heap-ordered if every element is less than or equal to its children
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A tree is heap-ordered if every element is less than or equal to its children.

Which of these is a binary heap?:

```
      2
     / \
    5   97
   / \  /  \
  40 7 99 40 7 99 40 7 99
```

```
      2
     / \  \
    5   97
   / \  / \
  40 7 99 40 7 99
```

```
      5
     / \
    2   97
   /  \
  40 7
```

```
      5
     / \
    2   97
   /  \
  40 7
```
Binary heap insertion

1. Add the new element at the end
2. Bubble up to restore invariant
Binary heap insertion

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Binary heap removal

1. Replace the root with the last element of the heap
2. Sink down to restore invariant
Binary heap removal

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The super cool thing about binary heaps

Instead of storing it as an actual tree with pointers:

```
2
\   \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \    \7```
The super cool thing about binary heaps

Instead of storing it as an actual tree with pointers:

A binary heap is stored in level-order in an array:
The super cool thing about binary heaps

Instead of storing it as an actual tree with pointers:

```
2
5
40
45
60
7
12
14
```

a binary heap is stored in level-order in an array:

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
2 5 6 40 7 4 90 45 60 12 14 75 8
```
The super cool thing about binary heaps

Instead of storing it as an actual tree with pointers:

```
  2
 /   \
 5    4
 /     /
40    7  6
  |    /   \
45   7    8
```

a binary heap is stored in level-order in an array:

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
2 5 4 40 7 6 90 45 60 12 14 75 8
```
Finding parents and children

Because the structure is *implicit*, we can’t just follow pointers

Suppose $i$ is the index of a node:

- How can we find its parent (if any)?
- How can we find its children (if any)?
Next time: another graph algorithm and another data structure to go with it