Amortized Time

CS 214, Fall 2019

Remember Union-Find?

We never said how much a single `union` or `find` operation costs

Instead, we said that m operations on n objects is $\mathcal{O}((m+n)\log^* n)$

Remember Union-Find?

We never said how much a single `union` or `find` operation costs

Instead, we said that m operations on n objects is $\mathcal{O}((m+n)\log^* n)$

This is because some long-running operations do maintenance that make other operations faster

Example: dynamic array

Dynamic Array ADT (1/2)

Looks like: $[3, 8, 2, 90, 5\rangle$

Signature:

```
interface DYN_ARRAY[T]:
    def len(self) -> nat?
    def get(self, index: nat?) -> T
    def set(self, index: nat?, element: T) -> NoneC
    def push(self, element: T) -> NoneC
    def pop(self) -> T
```

Dynamic Array ADT (2/2)

Laws:

$$\left\{a = \boxed{[v_1, \ldots, v_k\rangle}\right\}\ a.len() \Rightarrow k\ \{a = a_0\}$$

$$\left\{a = \boxed{[v_1, \ldots, v_k\rangle}\right\}\ a.push(w) \Rightarrow None\ \left\{a = \boxed{[v_1, \ldots, v_k, w\rangle}\right\}$$

$$\left\{a = \boxed{[v_0, \ldots, v_{k-1}, v_k\rangle}\right\}\ a.pop() \Rightarrow v_k\ \left\{a = \boxed{[v_0, \ldots, v_{k-1}\rangle}\right\}$$

$$\left\{a = \boxed{[\ldots, v_i, \ldots\rangle}\right\}\ a.get(i) \Rightarrow v_i\ \{a = a_0\}$$

$$\left\{a = \boxed{[\ldots, v_{i-1}, v_i, v_{i+1}, \ldots\rangle}\right\}$$

$$a.set(i, w) \Rightarrow None$$

$$\left\{a = \boxed{[\ldots, v_{i-1}, w, v_{i+1}, \ldots\rangle}\right\}$$

A naïve representation (1/2)

```
class DynArray[T] (DYN_ARRAY):
    let _data: VecC[T]

    def __init__(self):
        self._data = []

    def len(self):
        self._data.len()

    def get(self, index):
        self._data[index]

    def set(self, index, element):
        self._data[index] = element

    …
```

A naïve representation (2/2)

```
class DynArray[T] (DYN_ARRAY):
    …

    def push(self, val):
        let n = self.len()
        self._data = [ self._data[i] if i < n else val
                       for i in range(n + 1) ]

    def pop(self):
        let n = self.len()
        let val = self._data[n - 1]
        self._data = [ self._data[i]
                       for i in range(n - 1) ]
        return val
```

Naïve representation complexities

- *get*/*set*/*len* are $\mathcal{O}(1)$
- *push*/*pop* are $\mathcal{O}(n)$!

Naïve representation complexities

- *get*/*set*/*len* are $\mathcal{O}(1)$
- *push*/*pop* are $\mathcal{O}(n)$!

How long does it take to build an n–element array by *push*es?

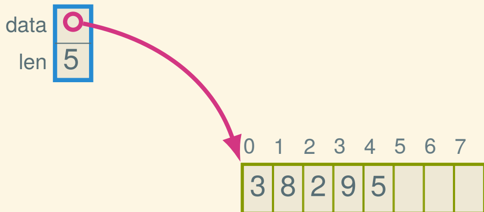
Naïve representation complexities

- *get*/*set*/*len* are $\mathcal{O}(1)$
- *push*/*pop* are $\mathcal{O}(n)$!

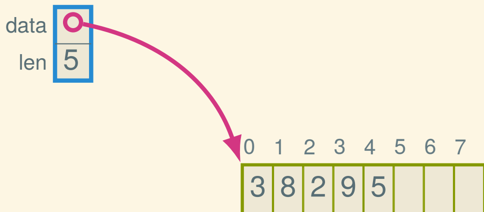
How long does it take to build an n–element array by *push*es?

$$\sum_{i=1}^{n} \mathcal{O}(i) = \mathcal{O}(n^2)$$

A better idea: leave extra space in the array



A better idea: leave extra space in the array



Note: DSSL2 vectors know their sizes and can tell you, but C pointers don’t know how many objects they point to, so in C you need to store the capacity yourself:

```
struct dyn_int_array
{
    int*   data_;
    size_t len_;  // number of elements
    size_t cap_;  // max `len_` without realloc
};
```

This is a *dynamic array*

It's called:

- `std::vector` in C++
- `ArrayList` in Java
- `list` in Python

Implementation (1/4)

```
class DynArray[T] (DYN_ARRAY):
    let _data: VecC[OrC(T, NoneC)]
    let _len:  nat?

    def __init__(self, initial_capacity: nat?):
        self._data = [None; initial_capacity]
        self._len  = 0

    def len(self):
        return self._len

    def capacity(self) -> nat?:
        return self._data.len()

    …
```

Implementation (2/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def get(self, index):
        self._bounds_check(index)
        return self._data[index]

    def set(self, index, element):
        self._bounds_check(index)
        self._data[index] = element
```

Implementation (2/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def get(self, index):
        self._bounds_check(index)
        return self._data[index]

    def set(self, index, element):
        self._bounds_check(index)
        self._data[index] = element

    def _bounds_check(self, index):
        if index >= self.len():
            error('DynArray: out of bounds')

    …
```

Implementation (3/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def pop(self):
        self._len = self._len - 1
        return self._data[self._len]
```

Implementation (3/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def pop(self):
        self._len = self._len - 1
        return self._data[self._len]

    # Avoids memory leaks:
    def pop(self):
        self._len = self._len - 1
        let result = self._data[self._len]
        self._data[self._len] = None
        return result

    …
```

Implementation (4/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def push(self, element):
        self._ensure_capacity(self._len + 1)
        self._data[self._len] = element
        self._len = self._len + 1
```

Implementation (4/4)

```
class DynArray[T] (DYN_ARRAY):
    …

    def push(self, element):
        self._ensure_capacity(self._len + 1)
        self._data[self._len] = element
        self._len = self._len + 1

    def _ensure_capacity(self, cap):
        if cap <= self.capacity(): return
        cap = max(cap, 2 * self.capacity())
        self._data = vec_copy_resize(cap, self._data)

    …
```

Time complexities

- *get*/*set*/*size* are $\mathcal{O}(1)$
- *pop* is $\mathcal{O}(1)$
- *push* is $\mathcal{O}(n)$ still

Time complexities

- *get*/*set*/*size* are $\mathcal{O}(1)$
- *pop* is $\mathcal{O}(1)$
- *push* is $\mathcal{O}(n)$ still

How long does it take to build an n–element array by *push*es?

Time complexities

- *get*/*set*/*size* are $\mathcal{O}(1)$
- *pop* is $\mathcal{O}(1)$
- *push* is $\mathcal{O}(n)$ still

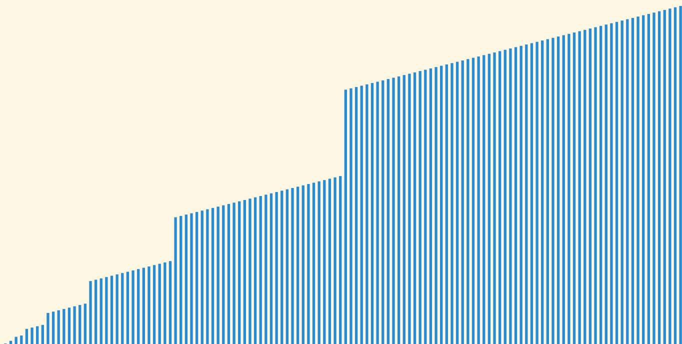
How long does it take to build an n–element array by *push*es?

$$\sum_{i=0}^{n} \mathcal{O}(i) = \mathcal{O}(n^2)?$$

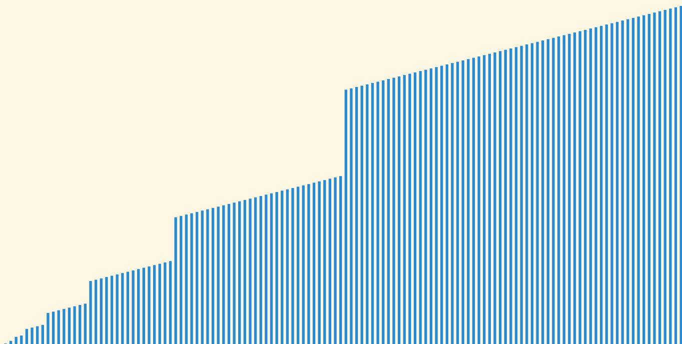
The peculiar thing about *push*

- Most of the time it's cheap
- Only occasionally do we need to grow (which is expensive):

Cumulative time



Cumulative time



It’s linear!

Dynamic array aggregate analysis

Suppose we create a new array and push n times. How can we show linear time?

Dynamic array aggregate analysis

Suppose we create a new array and push n times. How can we show linear time?

Let c_i be the cost of the ith insertion:

$$c_i = \begin{cases} i & \text{if } i - 1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Dynamic array aggregate analysis

Suppose we create a new array and push n times. How can we show linear time?

Let c_i be the cost of the ith insertion:

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

i	1	2	3	4	5	6	7	8	9	10
s_i	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1

Adding it up

Let $d_i = c_i - 1$ (the doubling cost)

Adding it up

Let $d_i = c_i - 1$ (the doubling cost)

Then,

$$\begin{aligned}
\sum_{i=1}^{n} c_i &= \sum_{i=1}^{n} (1 + d_i) \\
&= n + \sum_{i=1}^{n} d_i \\
&= n + \sum_{i=0}^{\log_2 n} 2^i \\
&= n + (n + \frac{n}{2} + \frac{n}{4} + \cdots) \\
&\leq 3n
\end{aligned}$$

Example: banker's queue (FIFO)

Banker's queue implementation (1/2)

```
class BankersQueue[T] (QUEUE):
    let front
    let back
    # Interpretation: the queue is the elements of
    # `front` in pop order followed by `back` in reverse

    def __init__(self, Stack: FunC[STACK!]):
        self.front = Stack()
        self.back = Stack()

    def len(self):
        self.front.len() + self.back.len()

    def empty?(self):
        self.front.empty?() and self.back.empty?()

    …
```

Banker's queue implementation (2/2)

```
class BankersQueue[T] (QUEUE):
    …

    def enqueue(self, element):
        self.back.push(element)
```

Banker’s queue implementation (2/2)

```
class BankersQueue[T] (QUEUE):
    …

    def enqueue(self, element):
        self.back.push(element)

    def dequeue(self):
        if self.front.empty?():
            if self.back.empty?():
                error('BankersQueue.dequeue: empty')
            while not self.back.empty?():
                self.front.push(self.back.pop())
        self.front.pop()
```

Banker's queue analysis (physicist style)

We assign a "potential" to each data structure state:

$$\Phi(q) = q\texttt{.back.len()}$$

Note that the potential of a new queue is 0, and the potential is never negative

Banker's queue analysis (physicist style)

We assign a "potential" to each data structure state:

$$\Phi(q) = q\texttt{.back.len()}$$

Note that the potential of a new queue is 0, and the potential is never negative

Then the amortized cost of an operation is

$$c + \Phi(q') - \Phi(q)$$

where c is the actual cost, q is the state before, and q' is the state after

Actual costs

Actual cost of enqueue operation: 1

Actual costs

Actual cost of enqueue operation: 1

Actual cost of cheap dequeue operation (when front isn't empty): 1

Actual costs

Actual cost of enqueue operation: 1

Actual cost of cheap dequeue operation (when front isn't empty): 1

Actual cost of expensive dequeue operation (with reversal) is the cost of the reversal (the number of elements reversed) plus the cost of a cheap dequeue: $n + 1$

Amortized cost of enqueue

- Actual cost of enqueue is 1
- Increases the length of the back by 1, hence $\Phi(q') - \Phi(q) = 1$

So amortized cost is $1 + 1 = 2$

Amortized cost of cheap dequeue

- Actual cost of cheap dequeue is 1
- No change in potential

So amortized cost is 1

Amortized cost of expensive dequeue

Let n be `q.back.len()`, the length of the back stack. Then:

- Actual cost is $n + 1$
- $\Phi(q) = n$ (before reversal)
- $\Phi(q') = 0$ (after reversal)

So amortized cost is $n + 1 + 0 - n = 1$.

Banker's queue operation worst-case time complexities

operation	single operation	amortized
enqueue	$\mathcal{O}(1)$	$\mathcal{O}(1)$
dequeue	$\mathcal{O}(n)$	$\mathcal{O}(1)$

Next time: random binary search trees