Random Binary Search Trees

CS 214, Fall 2019
The necessity of balance
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<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lceil \lg n \rceil$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>100</td>
<td>7</td>
</tr>
<tr>
<td>1,000</td>
<td>10</td>
</tr>
<tr>
<td>10,000</td>
<td>14</td>
</tr>
<tr>
<td>100,000</td>
<td>17</td>
</tr>
<tr>
<td>1,000,000</td>
<td>20</td>
</tr>
<tr>
<td>10,000,000</td>
<td>24</td>
</tr>
<tr>
<td>100,000,000</td>
<td>27</td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>30</td>
</tr>
</tbody>
</table>
An `rndbst?` (randomized BST of numbers) is either:
- None
- `_node(key?, nat?, rndbst?, rndbst?)`

```plaintext
let rndbst? = OrC(_node?, NoneC)
```

**struct** _node:
- let key: key?
- let size: nat?
- let left: rndbst?
- let right: rndbst?
Size maintenance

```python
def empty?(t: rndbst?) -> bool?:
    not _node?(t)

def size(t: rndbst?) -> nat?:
    t.size if _node?(t) else 0

def _fix_size(n: _node?) -> _node?:
    n.size = 1 + size(n.left) + size(n.right)
    n

def _new_node(k: key?) -> rndbst?:
    _node(k, 1, None, None)
```
Leaf insertion in DSSL2

The easy way to add elements to a tree—at the leaves:

def leaf_insert(t: rndbst?, k: key?) -> rndbst?:
    if empty?(t):
        return _new_node(k)
    elif k < t.key:
        t.left = leaf_insert(t.left, k)
        return _fix_size(t)
    elif k > t.key:
        t.right = leaf_insert(t.right, k)
        return _fix_size(t)
    else:
        return t
Leaf insertion
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```
7
/  \
3   11
/    /  \
1    9   13
/  \
0  2  4
```

Leaf insertion
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Leaf insertion
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?
The permutation distribution

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- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 — severely unbalanced (degenerate)
- 7, 3, 1, 0, 2, 5, 4, 6, 11, 9, 8, 10, 13, 12, 14 — balanced
- 7, 11, 3, 13, 9, 5, 1, 14, 12, 10, 8, 6, 4, 2, 0 — balanced

In fact, the only sequence to produce the right-branching degenerate tree is 0, …, 14

There are 21,964,800 sequences that produce the same perfectly balanced tree
Can we characterize how sequences of insertions produce (un)balanced trees?

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There are 21,964,800 sequences that produce the same perfectly balanced tree
A random BST tends to be balanced

If you generate a tree by leaf-inserting a random permutation of its elements, it will probably be balanced.

In particular, the expected length of a search path is

\[ 2 \ln n + \mathcal{O}(1) \]
A random BST tends to be balanced

If you generate a tree by leaf-inserting a random permutation of its elements, it will probably be balanced

In particular, the expected length of a search path is

$$2 \ln n + \mathcal{O}(1)$$

Unfortunately, we usually can’t do that, but we can simulate it
A tool: tree rotations

Note that order is preserved
In DSSL2

def _rotate_right(d):
    let b = d.left
    d.left = b.right
    b.right = d
    return b

def _rotate_left(b):
    let d = b.right
    b.right = d.left
    d.left = b
    return d
In DSSL2

def _rotate_right(d):
    let b = d.left
    d.left = b.right
    b.right = _fix_size(d)
    return _fix_size(b)

def _rotate_left(b):
    let d = b.right
    b.right = d.left
    d.left = _fix_size(b)
    return _fix_size(d)
def _k_in_n(k: nat?, n: nat?) -> bool?:
    return random(n) < k
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    return random(n) < k

\[
Pr\left[\_k\_in\_n(k, n)\right] = \frac{k}{n}
\]
Another random little helper

```python
def _k_in_n(k: nat?, n: nat?) -> bool?:
    return random(n) < k

Pr[_k_in_n(k, n)] = k/n
```

```python
def _one_in_n(n: nat?) -> bool?:
    return random(n) == 0

Pr[_one_in_n(n)] = 1/n
```
Root insertion

Using rotations, we can insert at the root:

- To insert into an empty tree, create a new node
- To insert into a non-empty tree, if the new key is greater than the root, then root-insert (recursively) into the right subtree, then rotate left
- By symmetry, if the key belongs to the left of the old root, root insert into the left subtree and then rotate right
def _root_insert(t: rndbst?, k: key?) -> rndbst?:
    if empty?(t):
        return _new_node(k)
    elif k < t.key:
        t.left = _root_insert(t.left, k)
        return _rotate_right(t)
    elif k > t.key:
        t.right = _root_insert(t.right, k)
        return _rotate_left(t)
    else:
        return t
Randomized insertion

We can now build a randomized insertion function that maintains the random shape of the tree:

- Suppose we insert into a subtree of size $k$, so the result will have size $k + 1$
- If the tree were random, the new element would be the root with probability $\frac{1}{k+1}$
- So we root insert with that probability, and otherwise recursively insert into a subsubtree
Randomized insertion in DSSL2

def insert(t: rndbst?, k: key?) -> rndbst?:
    if _one_in_n(size(t) + 1):
        return _root_insert(t, k)
    elif k < t.key:
        t.left = insert(t.left, k)
        return _fix_size(t)
    elif k > t.key:
        t.right = insert(t.right, k)
        return _fix_size(t)
    else:
        return t
To delete a node, we join its subtrees recursively, randomly selecting which contributes the root (based on size):
def _join(t1: rndbst?, t2: rndbst?) -> rndbst?:
    if empty?(t1): return t2
    elif empty?(t2): return t1
    elif _k_in_n(size(t1), size(t1) + size(t2)):
        t1.right = _join(t1.right, t2)
        return _fix_size(t1)
    else:
        t2.left = _join(t1, t2.left)
        return _fix_size(t2)
def delete(t: rndbst?, k: key?) -> rndbst?:
    if empty?(t):
        return t
    elif k < t.key:
        t.left = delete(t.left, k)
        return _fix_size(t)
    elif k > t.key:
        t.right = delete(t.right, k)
        return _fix_size(t)
    else:
        return _join(t.left, t.right)
Next: guaranteed balance