Self-Balancing Binary Search Trees

CS 214, Fall 2019

A self-balancing BST

Random binary search trees are *very likely* to be balanced

Self-balancing trees are *guaranteed* to be balanced

Balanced search tree survey

AVL trees

Due to: Georgy Adelson-Velsky & Evgenii Landis (1962)

Main idea: Maintain a *balance factor* giving the difference between each node's subtrees' heights

Local invariant: Balance factor between -1 and 1, maintained via rotations

Global invariant: Tree is approximately height-balanced

2–3 trees

Due to: John Hopcroft (1970)

Main idea: 2-nodes have one element and two children;
3-nodes have two elements and three children

Local invariant: All subtrees of a node have the same height

Global invariant: Every leaf is at the same depth

Advantage: Faster insertions, slower lookups (compared to AVL)

B-trees

Due to: Rudolf Bayer & Ed McCreight (1971)

Main idea: Generalizaton of 2–3 trees up to *k* children.

Local invariant: Like 2–3 trees, but allow up to $k/2$ missing children.

Global invariant: Every leaf is at the same depth

Use: On-disk databases (or modern memory hierarchies)

Advantage: Larger nodes means fewer disk accesses (or cache misses)

2–3–4 trees (a/k/a 2–4 trees)

Due to: Rudolf Bayer (1972)

Main idea: B-tree of order 4.

Why interesting: *Isometry* of *red–black tree*

Red–black trees

Due to: Leonidas J. Guibas & Robert Sedgewick (1978)

Main idea: Every node has an extra bit marking it “red” or “black”

Local invariant: No red node has a red parent

Global invariant: Equal number of black nodes from root to every leaf

Advantage: Faster insertions, slower lookups (compared to AVL); easier representation than 2–3(–4) trees

Splay trees (randomized or amortized!)

Due to: Daniel Sleator & Robert Tarjan (1985)

Main idea: Cache recently accessed elements near the root of the tree

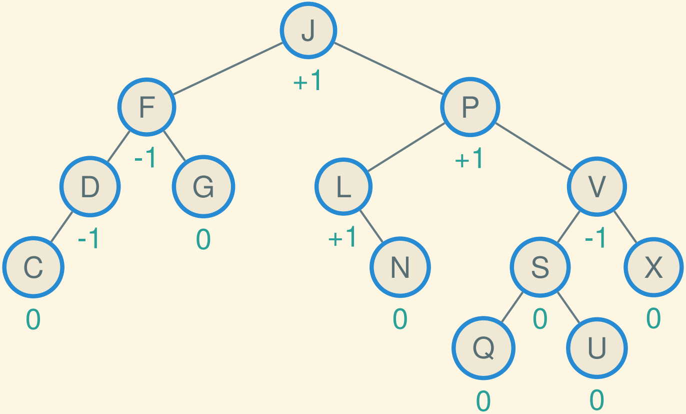
Local invariant: *Complicated; required amortized analysis*

Global invariant: Paths are *very likely* to be $\mathcal{O}(\log n)$

Advantage: Self optimizing; no extra balance data

AVL trees

Example of an AVL tree



Local invariant maintains global property

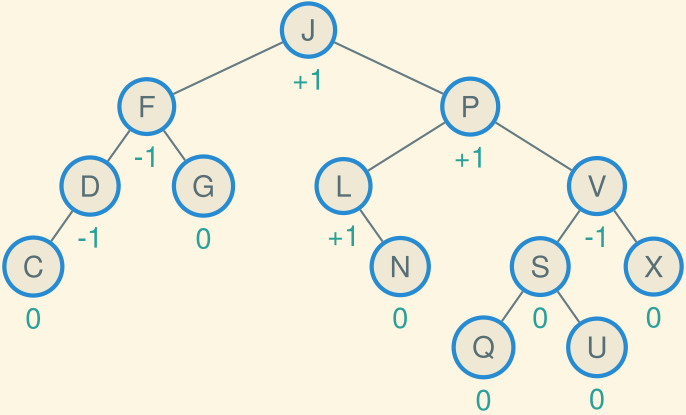
- Balance factors are maintained locally
- Never recompute them from scratch
- Yet the whole tree stays reasonably balanced

AVL insertion

- First do a normal leaf insertion
- Track balance factors on the way back up to the root
- Adjust with rotations as necessary

AVL insertion example

Let's insert H:



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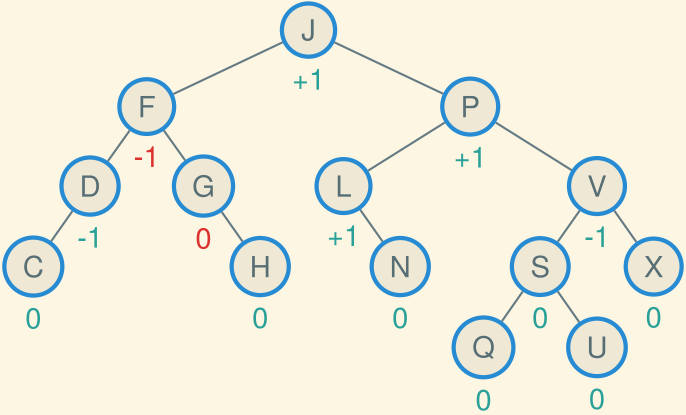

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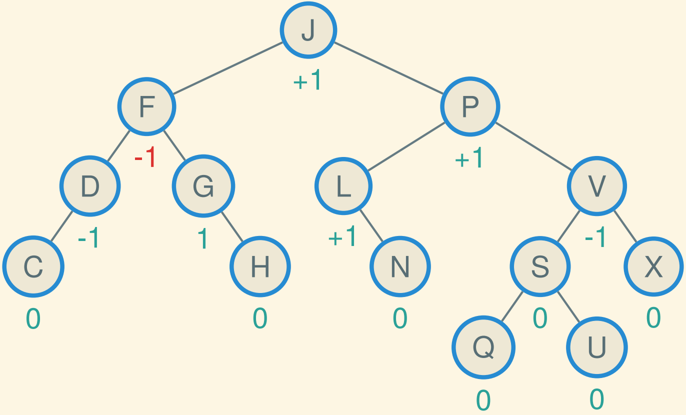

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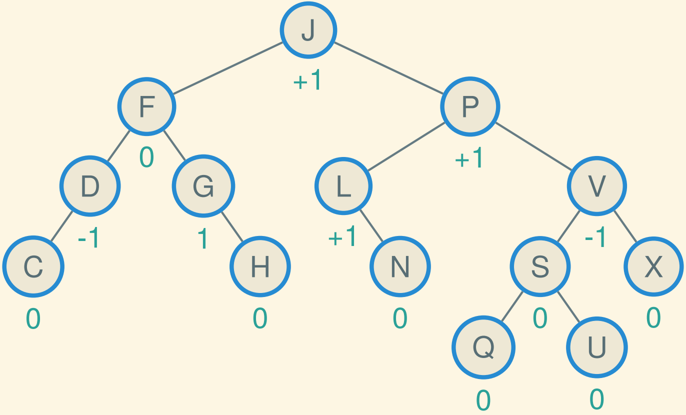
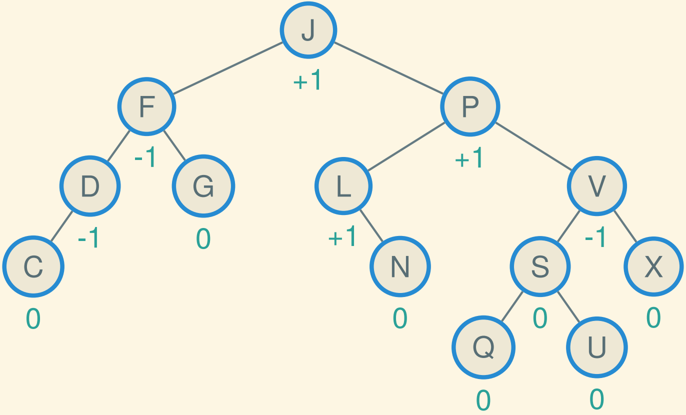


Another AVL insertion example

Let's insert B:



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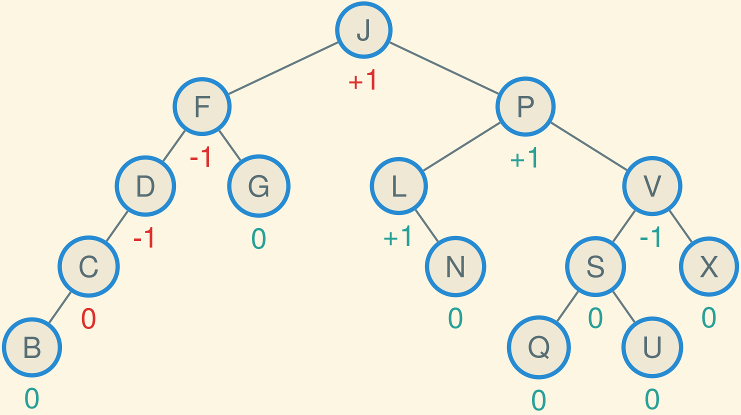

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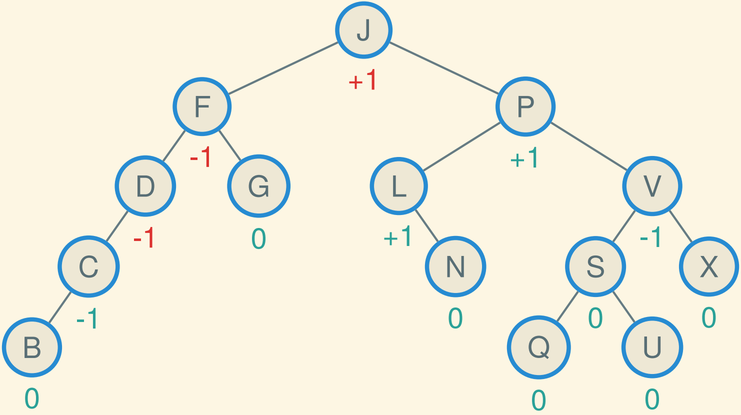

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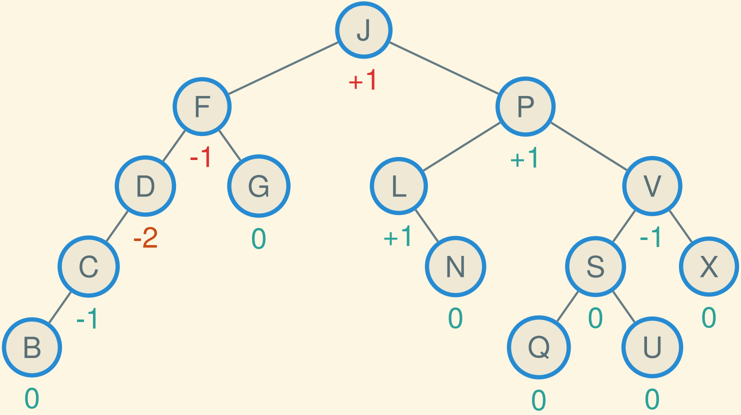

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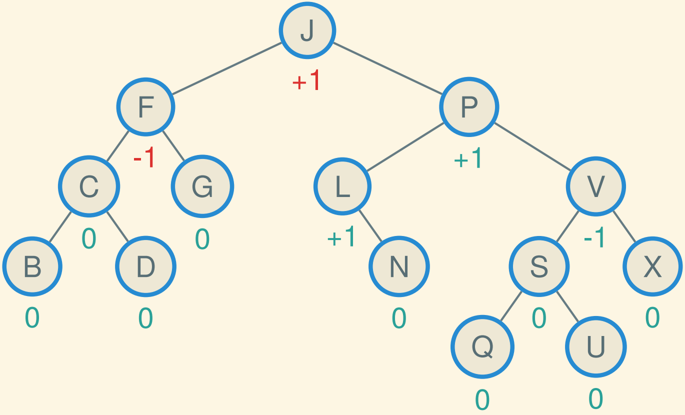

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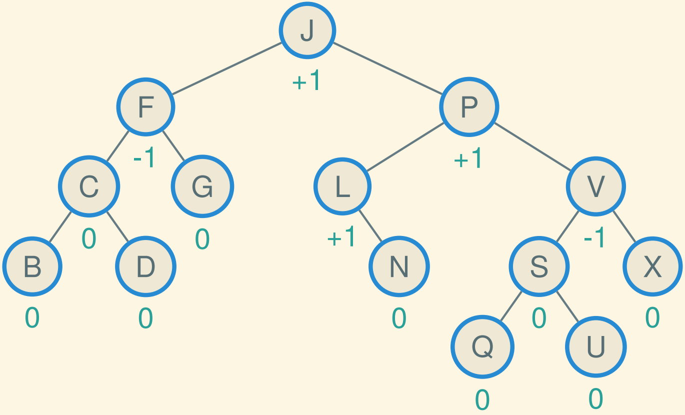
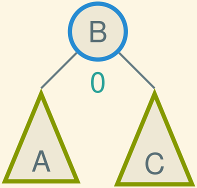


Maintaining the AVL property

Suppose we have an AVL tree:



(Convention: triangles represent equal-height subtrees.)

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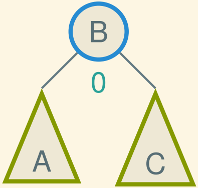
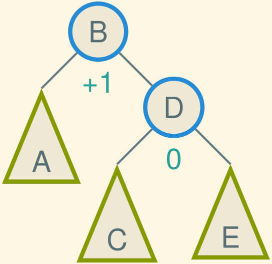


(Convention: triangles represent equal-height subtrees.)

Right now the balance factor is 0. So if we insert into A or C and that subtree grows in height, it becomes -1 or 1.

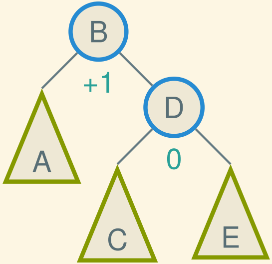
Maintaining the AVL property



Right now the balance factor at B is +1.

Suppose we insert into A. What happens to B's balance factor?

Maintaining the AVL property

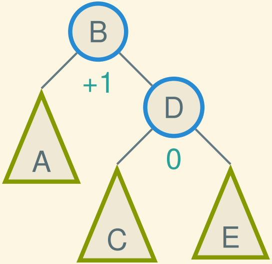


Right now the balance factor at B is +1.

Suppose we insert into A. What happens to B's balance factor?

- If no change in A's height then no change in B's balance
- If A's height grows then B's balance factor goes to 0

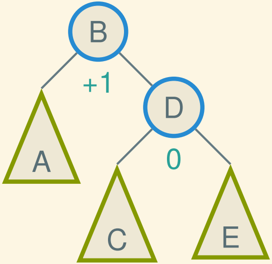
Maintaining the AVL property



Right now the balance factor at B is +1.

Suppose we insert into C. What happens to B's balance factor?

Maintaining the AVL property

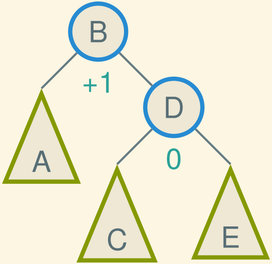


Right now the balance factor at B is +1.

Suppose we insert into C. What happens to B's balance factor?

- If no height change then B's balance doesn't change
- If C grows then B's balance factor becomes +2

Maintaining the AVL property



Right now the balance factor at B is +1.

Suppose we insert into C. What happens to B's balance factor?

- If no height change then B's balance doesn't change
- If C grows then B's balance factor becomes +2—not okay!

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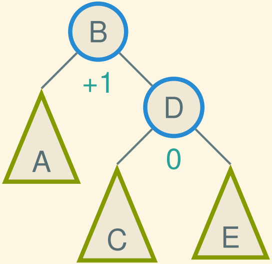


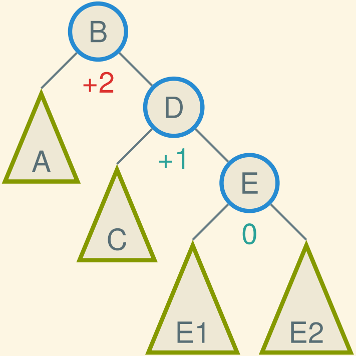
Right now the balance factor at B is +1.

Likewise, suppose we insert into E. What happens to B's balance factor?

- If no height change then B's balance doesn't change
- If E grows then B's balance factor becomes +2—not okay!

The right-right case

If the height of the right-right subtree (E) increases, we get a situation like this:



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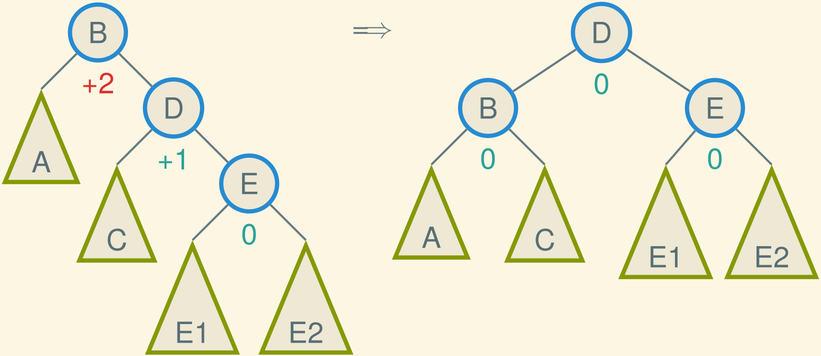
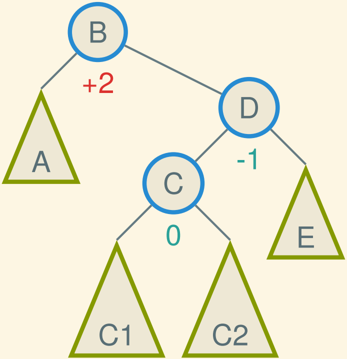


The right-left case

If the height of the right-left subtree (C) increases, we get a situation like this:



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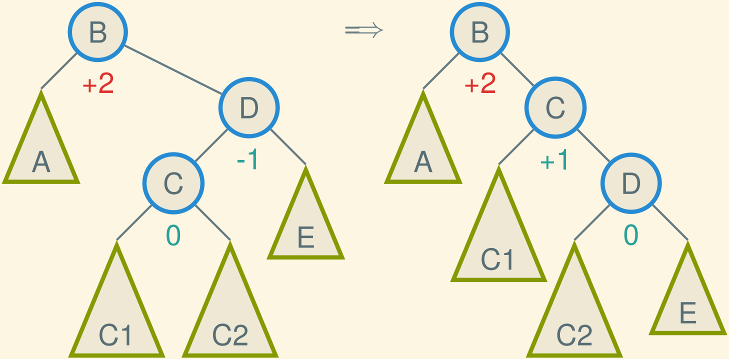

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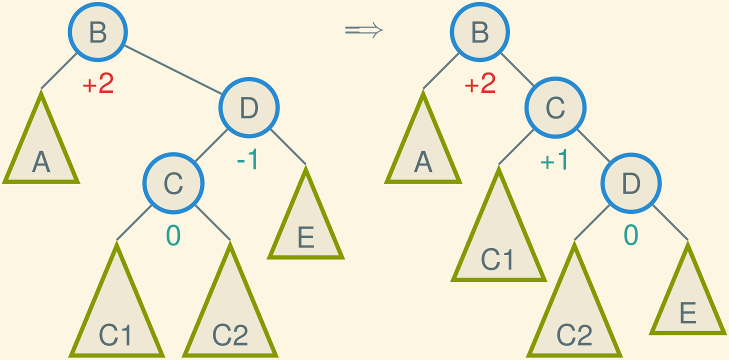

But this is now the right-right case, which we know how to handle!

Maintaining the AVL property

- We've seen the right-right and right-left cases
- The left-left and left-right cases are symmetrical
- Deletion is like ordinary BST deletion, with the same rebalancing cases

See `avl.rkt`.

Red–black trees

The red–black tree rules

The rules:

1. Nodes are colored red or black.

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4. Every red node has a black parent.

The red–black tree rules

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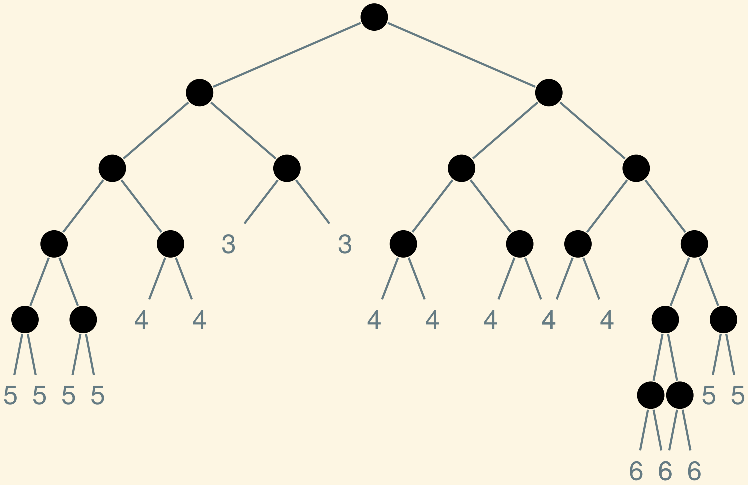
1. Nodes are colored red or black.
2. The root is always black.
3. “Dummy leaves” are black.
4. Every red node has a black parent.
5. For every node, all paths to leaves have the same “black height.”

Red–black colorability



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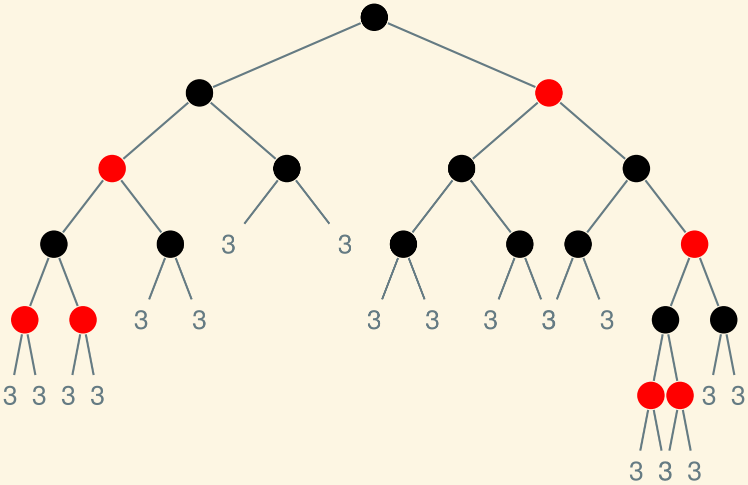

Red–black tree insertion

1. Leaf insert, like any other BST

Red–black tree insertion

1. Leaf insert, like any other BST
2. Color new node red.

Red–black tree insertion

1. Leaf insert, like any other BST
2. Color new node red.
3. If parent is also red (violating rule 4), color parent black and look for problems further up.

Next: C and C++