Amortized Analysis

EECS 214

November 11–13, 2015

Take-aways

- What is *amortized time*?
- How does amortized time differ from *average time*?
- When is amortized time useful, and when might we want to avoid it?
- How can we figure out the amortized time of data structure operations?
- How does a dynamic array achieve its amortized time complexity?

Example: dynamic arrays

Language	**Type**
C++	`std::vector`
Java	`ArrayList`
Python	`list`
Ruby	`Array`

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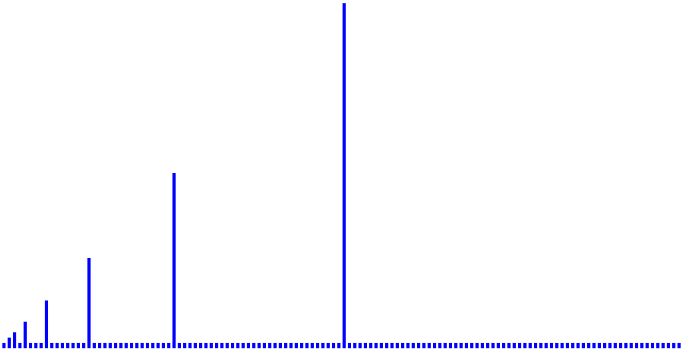
ASL	*you’re on your own*

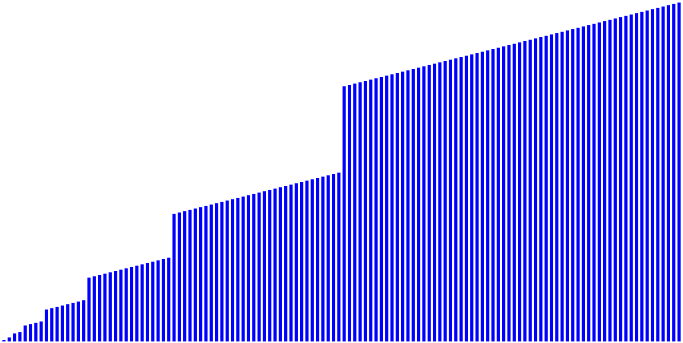
Iteratively growing a dynamic array

```
std::vector<int> v;
for (int i = 0; i < N; ++i) v.push_back(i);

ArrayList<Integer> v = new ArrayList<>();
for (int i = 0; i < N; ++i) v.add(i);

v = list()
for i in range(0, 10): v.append(i)

v = Array.new
for i in 0 ... N do v.push(i) end
```

Time per operation



Accumulated time



What's it doing?

- A dynamic array is backed by a fixed-size array with excess capacity:
  `(define-struct dynarray [data size])`
- When the array fills, allocate a fixed-size array that's twice as big and copy over the elements.

Time complexity of a single insertion

A single insertion:

$$T_{\text{insert}}(n) = \mathcal{O}(n)$$

Time complexity of a sequence of insertions

Hence, for a sequence of insertions:

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&= \mathcal{O}\left(\frac{m(m+1)}{2}\right) \\
&= \mathcal{O}(m^2)
\end{aligned}
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Amortized time complexity

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Amortization is about the *worst case*, not merely the *average case*.

Banker's method: real costs vs. accounting costs

Let c_i be the actual cost of the ith operation
Let c'_i be the charged cost of the ith operation

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Let c_i be the actual cost of the ith operation
Let c_i' be the charged cost of the ith operation—we choose this!

If total actual cost does not exceed the total charged cost,

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c_i' ,$$

then we say that the ith operation has worst-case *amortized* time $\mathcal{O}(c_i')$,

Amortized time for dynamic array insertion (banker style)

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i	1	2	3	4	5	6	7	8	9	10

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c'_i	1	1	1	1	1	1	1	1	1	1

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c_i	1	2	3	1	5	1	1	1	9	1
${c'}_i$	1	1	1	1	1	1	1	1	1	1
bal_i	0	-1	-1	-1	-1	-1	-1	...		

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bal_i	2	3	3	5	3	5	7	9	3	5

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bal_i	5	8	8	14	8	14	20	26	8	14

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Physicist's method: potential "energy"

We define a potential function Φ on data structure states, where:

$$\Phi(v_0) = 0 \qquad \text{starts at } 0$$

$$\Phi(v_t) \geq 0 \qquad \text{never goes negative}$$

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- ✓ The initial vector has no size and no capacity, so $\Phi(v_0) = 0$
- ✓ The capacity is never more than twice the size, because we double when it's full; hence $2n \geq m$; hence $\Phi(v) = 2n - m \geq 0$.

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$$c_i' = 1 + (2(n+1) - m) - (2n - m)$$

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Let's compute c'_i for insertion. Remember that $c'_i = c_i + \Phi(v_i) - \Phi(v_{i-1})$. There are two possibilities:

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- ✓ If $n = m$ then $c_i = n + 1$ (copy plus simple insert). So

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Another example: (naïve) persistent banker's queue

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What if we want a persistent FIFO queue with sub-linear operations?

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