CS 395/495-26: Spring 2002

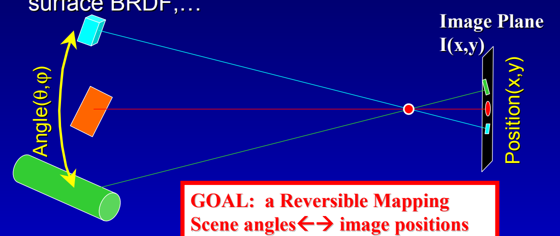
IBMR: Week 2 A
2-D Projective Geometry

Jack Tumblin
jet@cs.northwestern.edu

Recall: Scene & Image

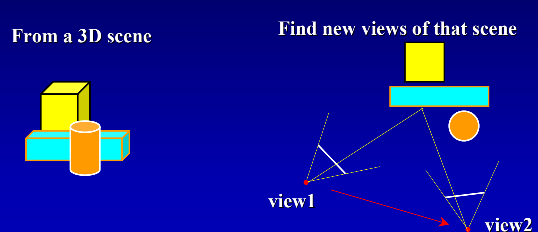
Light + 3D Scene:
Illumination, shape, movement, surface BRDF,…

2D Image:
Collection of rays through a point



View Interpolation: How?

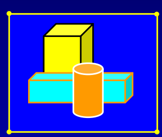
- Chapter 2: 3D Projection (soon)



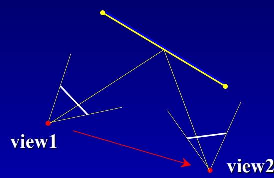
View Interpolation: How?

- **BUT FIRST,** the simpler case
  Chapter 1: 2D Projection

From a flat 2D image,



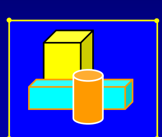
Find new views of that image



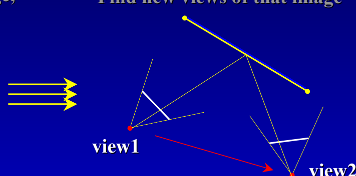
Answer: 2D Homogeneous Coords

Chapter 1: 2D Projection

From a flat 2D image,



Find new views of that image



Cartesian (x,y) coordinates in R^2 ⇛ **2D Homogeneous (x_1 x_2 x_3) coordinates in P^2**

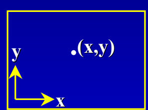
Overview

- Project 1: 1: Image-Colored Mesh Viewer
- Chapter 1: <u>any</u> 2D plane, as seen by
  <u>any</u> 3D camera view (texture mapping generalized)
- Homogeneous Coordinates are Wonderful
- Everything is a Matrix:
  - Conics (you can skip for now)
  - Aside: interpolation of pixels and points…
- Transform your Point of View: **H** matrix
- Parts of H: useful kinds of Transforms

2D Homogeneous Coordinates

- WHY? makes **MUCH** cleaner math!
  - Unifies lines and points
  - Puts perspective projection into matrix form
  - No divide-by-zero, lines at infinity defined…

in $\mathbf{R}^2$, write point $\mathbf{x}$ as $\begin{bmatrix} x \\ y \end{bmatrix}$



But in $\mathbf{P}^2$, write same point $\mathbf{x}$ as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where:

$x = x_1 / x_3$,
$y = x_2 / x_3$,
x_3 = anything non-zero!
(but usually defaults to 1)

Homogeneous Coordinates

WHAT?! Why x_3? Why 'default' value of 1?

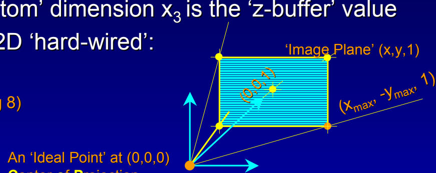
- Look at lines in $\mathbf{R}^2$:
  - 'line' == all (x,y) points where $ax + by + c = 0$
  - scale by 'k' →→ no change: $kax + kby + kc = 0$
- Using 'x_3' for points **UNIFIES** notation:
  - line is a 3-vector named $\mathbf{l}$
  - now point (x,y) is a 3-vector too, named $\mathbf{x}$

$ax + by + c = 0$ $\quad \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}$ $\quad \mathbf{x}^T \mathbf{l} = \mathbf{0}$

Useful 3D Graphics Ideas

- Every Homog. point (x_1,x_2,x_3) describes a 3D ray
- 'Phantom' dimension x_3 is the 'z-buffer' value
- 3D→2D 'hard-wired':

(Fig 1.1, pg 8)



- Homogeneous coords allows translation matrix:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+3 \\ y+5 \\ 1 \end{bmatrix}$$

2D Homogeneous Coordinates

Important Properties 1 (see book for details)

- 3 coordinates, but only 2 degrees of freedom (only 2 ratios (x_1 / x_3), (x_2 / x_3) can change)
- DUALITY: points, lines are interchangeable
  - Line Intersections = point: $\mathbf{l}_1 \times \mathbf{l}_2 = \mathbf{x}$
  - Point 'Intersections' = line: $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{l}$ (a 3D cross-product)
  - Projective theorem for lines ←→ theorem for points!

Homogeneous Coordinates

Important Properties 2 (see book for details)

- Neatly Sidesteps 'divide-by-zero': in (x,y) space (R^2)
  - Store $(x_1, x_2, x_3)^T$, compute (x_1 / x_3), (x_2 / x_3) **only** if OK.
  - Define 'Ideal Point' $(x,y,0)^T$ as a point at infinity in (x,y) space (R^2)
    - Note! (x,y,0) is an entire plane of points in the 3D $(x_1, x_2, x_3)^T$ xspace
    - Note! 'Center of Projection' is an ideal point. (Image plane 'wraps around' to that point?)
  - Define 'Line at infinity' $\mathbf{l}_\infty \equiv (0,0,1)^T$, or "0x +0y +1=0"
    - All ideal points are on $\mathbf{l}_\infty$: proof? $\mathbf{l}_\infty \cdot (x_1, x_2, 0)^T = 0$
    - All parallel lines Let $\mathbf{l} = (a,b,c)^T$ and $\mathbf{l}' = (a,b,c')^T$.
    - Any line $\mathbf{l}$ intersects with $\mathbf{l}_\infty$ line at an ideal point
    - Two parallel lines $\mathbf{l}$ and $\mathbf{l}'$ always meet at an ideal point (page 7)

Homogeneous Coordinates

Important Properties 3 (see book for details)

- Conic Sections (in the (x,y) plane, a.k.a. R^2)
  - SKIP this until a little later …
  - Core idea: Conics are Well-Behaved in the upcoming view-interpolations
    - Elegant homogeneous matrix form for any and all conic curves (ellipse, circle, parabola, hyperbola, degenerate lines & points):

      $\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$
    - Find any conic curve from 5 (x,y) points on the curve
    - Nice dual form exists too (analogous to line/point duality)!

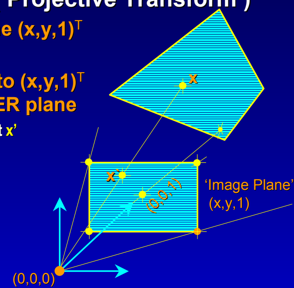
Homogeneous Coordinates

Important Properties 4 (see book for details)

- View Interpolation (‘Projective Transform’)
  - ‘Central’ image plane $(x,y,1)^T$
    - Choose a known point x’
  - Apply 3x3 Matrix H to $(x,y,1)^T$ to make some OTHER plane
    - Ray through known point x’ pierces unknown point x

H x’ = x

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Projective Transform H

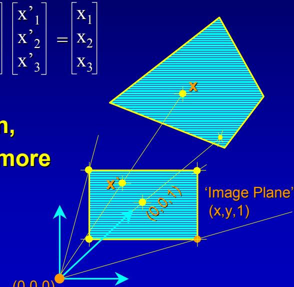
H matrix: Plane-to-Plane mapping

H x’ = x

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What if H is unknown, but we have four or more pairs of x,x’ points? (see pg 15)

x_1, x'_1, x_2, x'_2
$x_3, x'_3, x_4, x'_4 \ldots$



Projective Transform H

Finding H from point pairs (correspondences)

- We know that **Hx’ = x**, and
- we know at least 4 point pairs x’ and x that satisfy it:
- **ATTEMPT 1:** ‘plug & chug’ make a matrix of **x** and x’ values…

H x’ = x

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{H} \end{bmatrix} \begin{bmatrix} x'_1 & x'_2 & x'_3 & x'_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

Projective Transform H

Finding H from point pairs (correspondences)

- We know that **Hx' = x**, and
- we know at least 4 point pairs x' and x that satisfy it:
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$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

UNKNOWN!

$$\begin{bmatrix} \mathbf{H} \end{bmatrix} \begin{bmatrix} x'_1 & x'_2 & x'_3 & x'_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

END