CS 395/495-26: Spring 2002

IBMR: Week 3 A

2D Projective Geometry

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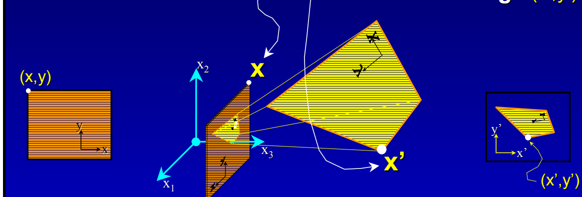
Projective Transform H

2D image (x,y)
→→
Homog. coords
$[x,y,1]^T$ = **x**

Apply the 3x3 matrix H
Hx = x'

Homog. coords
x' = $[x',y',1]^T$
→→
2D image (x',y')



The bits and pieces of H

H has 8 independent variables (DOF)

- Computer Graphics method (3x3 matrix):
  - 2D Translation (t_x, t_y)
  - 3D Scale (s_x,s_y,s_z)
  - 3D Rotation (θ_x,θ_y,θ_z)
- Computer Vision method (2D projective):
  - Isometry--3DOF(2D translate t_x,t_y; 2Drotate θ_z;)
  - Similarity--4DOF (add uniform scale s;)
  - Affine --6DOF (add orientable scale s_θ/s, $s_{\theta\perp}$/s)
  - Projective--8DOF (changes x_3; 3D-rotation-like)

Image Rectifying: Undo parts of H

$x' = H\,x$ where $H = H_S\,H_A\,H_P$

GOAL: Put world plane x' into view plane

- Affine Rect.; (find only H_P (2DOF));
- Metric Rect.; (find H_A and H_P (6DOF));
- Full Rect.; (find all of H (8DOF));

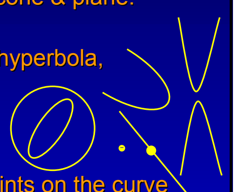
METHODS:

1. Affine: 'Vanishing Point', Infinity line methods
2. Metric: Conics & Circular Points
3. Full: 4-point correspondence

Conic Review

Review conics first (pg.8)

- **'Conics'** == intersection of cone & plane:
- Many possible shapes:
  circles, ellipses, parabola, hyperbola,
  degenerate lines & points



Dual Forms:

- 'Point Conics' defined by points on the curve
- 'Line Conics' defined by lines tangent to curve

Point Conics: C

- Equation of any/all conics solve a 2D quadratic:
  $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- Write in homogeneous coordinates:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

- C is symmetric, 5DOF (because x_3 scaling)
- Find any **C** from 5 homogeneous points

hence a 'Point Conic'

(solve for null space—see book pg 9)

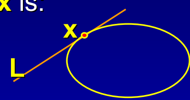
Point Conics: C

- Matrix **C** makes conic curves from points **x**:

  $\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$ **C** is a **'point conic'**

- The tangent line **L** for point **x** is:

  $\mathbf{L} = \mathbf{C}\,\mathbf{x}$



- Projective transform by H is:

  $\mathbf{C}' = \mathbf{H}^{-T}\,\mathbf{C}\,\mathbf{H}^{-1}$

(messy!)

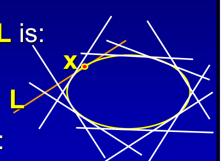
'Dual' or Line Conics: C

- Matrix **C*** makes conic curves from lines **L**:

  $\mathbf{L}^T\,\mathbf{C}^*\,\mathbf{L} = 0$ **C** is a **'line conic'**

- The tangent point **x** for line **L** is:

  $\mathbf{x} = \mathbf{C}^*\,\mathbf{L}$



- Projective transform by H is:

  $\mathbf{C}^{*\prime} = \mathbf{H}\,\mathbf{C}^*\,\mathbf{H}^T$

- If **C** is non-singular (rank 3), then $\mathbf{C}^* = \mathbf{C}^{-1}$

Degenerate Conics

$\mathbf{x}\,\mathbf{C}\,\mathbf{x} = 0$ or $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

- General form of point conic:

  $ax^2 + bxy + cy^2 + dx + ey + f = 0$ (or)

  $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + fx_3 = 0$

- A special 'degenerate' case, 'infinite radius circle' as $x_3 \rightarrow 0$

  $1x_1^2 + bx_1x_2 + 1x_2^2 + dx_1 + ex_2 + 0x_3 = 0$

$\mathbf{C}_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ solution: 2 complex **x** vectors: $\begin{bmatrix} \mathbf{1} \\ \mathbf{i} \\ \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{1} \\ \mathbf{-i} \\ \mathbf{0} \end{bmatrix}$

Conic Weirdness I

'Circular Points' **I**= $\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$, **J**= $\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$; or $\mathbf{C_\infty} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- 'Ideal points'; 2 points at infinity / horizon, but…
- **Complex :** real part on **x** axis, imag. on **y**
- "Where infinite circle hits infinity line L_∞ [0,0,1]"
- Where? imaginary! (infinite x,y axes, maybe?)
- Affected by H_P only. (H_S, H_A have no effect)

?WHY BOTHER?

- 3 real pts + 2 circular points → find **circle** conic
- To measure **angles** in projective space

Conic Weirdness 2

'Circular Points' **I**= $\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$, **J**= $\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$; and $\mathbf{C_\infty} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- $\mathbf{C_\infty}$ is a **point** conic, affected by H_P only.
- $\mathbf{C^*_\infty}$ is **line** conic, but the **same matrix** (!?!?!)
  - Is same as outer product: $\mathbf{IJ^T + JI^T}$
  - Transform this line conic as: $\mathbf{C^*_\infty{}' = H\, C^*_\infty\, H^T}$
  - Affected by H_P and H_A only, (H_S does nothing)
  - Includes L_∞ the infinity line ($L_\infty C^*_\infty L_\infty$=0)

ALL transforms of $\mathbf{C^*_\infty}$ have only 4DOF (pg 33)

Conics: Angle Measuring

- Matrix H transforms $\mathbf{C^*_\infty}$ to another space:

$$\mathbf{C^*_\infty{}' = H\, C^*_\infty\, H^T}$$

Angles:

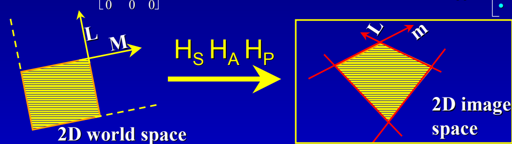
- define 'world space' $\mathbf{C^*_\infty}$ as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- (for any two world-space homogeneous lines) **L** and **m** are **perpendicular** iff $\mathbf{L^T\, C^*_\infty\, m = 0}$
- Also true for <u>transformed</u> **L, m** and $\mathbf{C^*_\infty}$, so…
- $\mathbf{C^*_\infty{}'}$ in image space→angle θ in world space

$$\cos(\theta) = \frac{(L^T\ C^*_\infty{}'\ m)}{\sqrt{(L^T C^*_\infty{}'\ L)(m^T C^*_\infty{}'\ m)}}$$

Undoing H: Metric Rect. 1

- **C^*_∞'** has only 4DOF (before,after **ANY** H)
- **All C^*_∞'** DOF are contained in (H_A H_P)
- **SVD can convert C^*_∞' to** (H_A H_P) matrix!
- **Task is then** 'find C^*_∞' to find H'

$$C^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C^*_\infty{}' = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 \end{bmatrix}$$



Undoing H: Metric Rect. 1

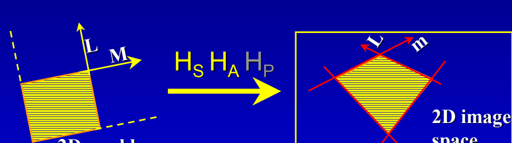
- **$H = H_S H_A H_P$ =** $\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix}$
- Book shows how to write (simple, tedious):

$$C^*_\infty{}' = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix}$$

where K is 2x2 symmetric (affine part: 2DOF)
v is 2x1 vector, (projective part: 2DOF)

Undoing H: Metric Rect. 1

- $C^*_\infty{}' = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix} \rightarrow \begin{bmatrix} KK^T & 0 \\ 0 & 0 \end{bmatrix}$
- First, ignore projective part: set v=0.
- Choose two pairs of perpendicular lines L,m
- Use **L^{T}' C^*_∞' m' = 0** to solve for K



Matrix Math & SVDs

- Matrix multiply is a change of coordinates
  - Rows of A = coordinate axis vectors
  - Ax = a dot product for each axis
- input space→output space
- SVD ‘factors’ matrix A into 3 parts:
  - U = orthonormal basis

svd(A) = U S V

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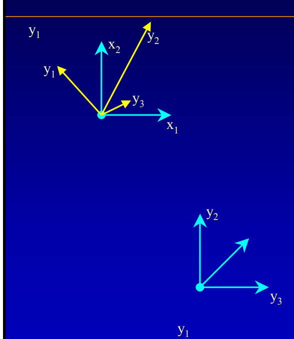
SVD Review



Undoing H: Metric Rectification 2

END