CS 395/495-26: Spring 2002

IBMR: Week 3 B

SVD Review, & Finish 2D Projective Geometry

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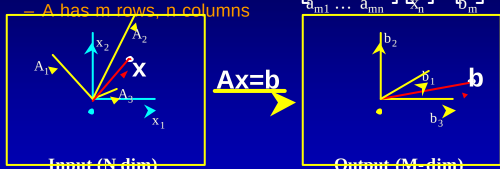
SVD Review: What is it?

- Matrix Multiply: Ax = b
  - x and b are column vectors
  - A has m rows, n columns

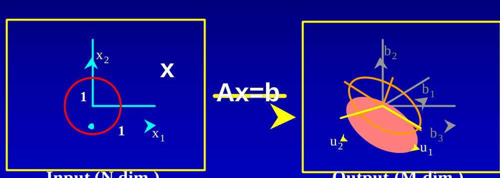
$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$



- N-dim. input space→M-dim. output space
  - Rows of A = new coordinate axes
  - Ax = a dot product for each new axis

SVD Review: What is it?

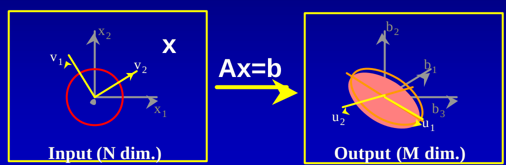
- Sphere of all unit-length **x** → output ellipsoid
- Its axes form orthonormal basis vectors U_i:



- SVD(A) = **U**SV^T columns of U = output basis vectors

SVD Review: What is it?

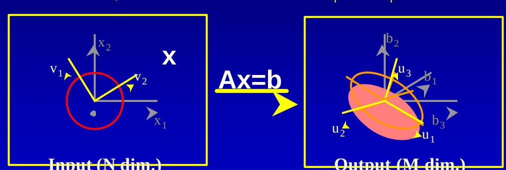
- For each U_i, make a matching V_i that:
  - Transforms to U_i (with scaling s_i) : $s_i (A V_i) = U_i$
  - Forms an orthonormal basis of input space



- SVD(A) = USV^T columns of V = input basis vectors

SVD Review: What is it?

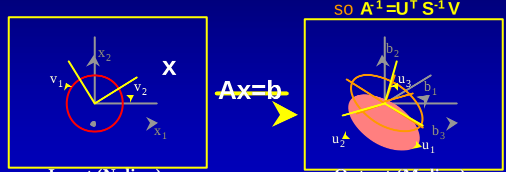
- Finish: SVD(A) = USV^T
  - add 'missing' U_i or V_i, define s_i=0.
  - Singular matrix **S**: diagonals are s_i
  - Matrix **U**, Matrix **V** have columns U_i and V_i



A = USV^T 'Find Input & Output Axes, Linked by Scale'

'Let SVDs Explain it All for You'

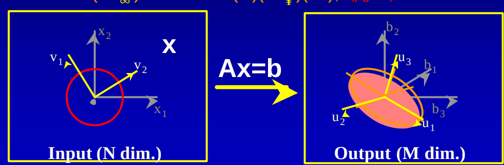
- Orthonormal U and V: $U^{-1} = U^T$, $V^{-1} = V^T$
- Rank(A)? # of non-zero singular values s_i
- 'ill conditioned'? Some s_i are nearly zero
- 'Invert non-square A' ? A=USV^T; $U^T S^{-1} V A = I$; so $A^{-1} = U^T S^{-1} V$



A = USV^T 'Find Input & Output Axes, Linked by Scale'

‘Let SVDs Explain it All for You’

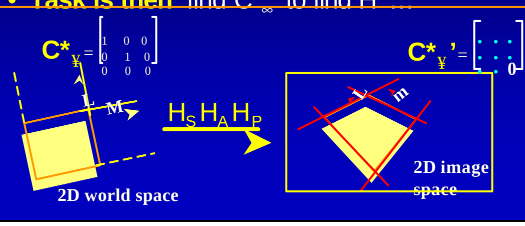
– Solve **Ax=0** (Null Space of x): Columns V_i whose $s_i=0$ span all solutions, so write $x = aV_{i1} + bV_{i2} + \ldots$
– Symmetry: if $A^T = A$ then SVD(A) is too: $A = USU^T$
– $C^*_\infty{}' = H\, C^*_\infty H^T$ is symmetric, so SVD finds H for you! $SVD(C^*_\infty{}') = USU^T = (H)(C^*_\text{¥})(H^T)$, (pg 35)



$A = USV^T$ ‘Find Input & Output Axes, Linked by Scale’

Undoing H: Metric Rectification

- $C^*_\text{¥}{}'$ has only 4DOF (before,after **ANY** H)
- **All** $C^*_\text{¥}{}'$ DOF are contained in $(H_A\ H_P)$
- **SVD can convert** $C^*_\text{¥}{}'$ **to** $(H_A\ H_P)$ matrix!
- **Task is then** ‘find $C^*_\infty{}'$ to find H’ …



? But how do we find $C^*_\text{¥}{}'$?

One Answer: use perpendicular lines

– (Recall) we defined ‘world space’ $C^*_\text{¥}$ as $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

– Then: 2 world-space lines **L** and **m** are **perpendicular** iff $L^T C^*_\text{¥}\, m = 0$

– Also true for transformed **L, m** and $C^*_\text{¥}$

– **H** transforms $C^*_\text{¥}$ to image space $C^*_\text{¥}{}'$ by: $C^*_\text{¥}{}' = H\, C^*_\text{¥}\, H^T$

– Now split up H and try it: $H = H_S H_A H_P$, so…

Undoing H: Metric Rectification

- (recall) $H = H_S\, H_A\, H_P = \begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v \end{bmatrix}$
- Tedious algebra (pg35) shows symmetry:

$$C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix}$$

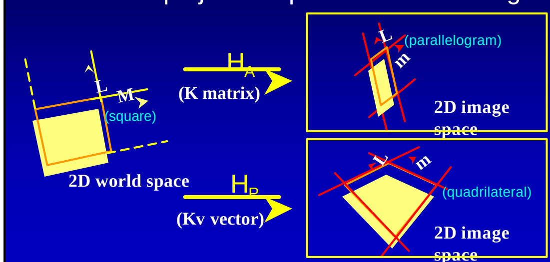
where K is 2x2 symmetric (affine part: 2DOF)

v is 2x1 vector, (projective part: 2DOF)

- ?But what do K and V really control?

Compare H_A and H_P

- HA is '2D skew': directional scaling:
- HP is '3D projection': parallel lines converge



Undoing H: Metric Rectification

- OK, then how do we find K and v?

$$C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix}$$

- Choose known perpendicular line pairs (L_i, m_i), then compute by:
  - Method 1a: (pg 36) Assume v=0, solve for K
  - Method 1b: (NOT in book) Assume k=0, solve for v
  - Method 2: Rearrange, solve for full $C^*_{\infty}{}'$ then get H using SVD.

Undoing H: Metric Rect. 1a

Method 1a: (pg 36)   Assume v=0, solve for K

$$C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix} \rightarrow \begin{bmatrix} KK^T & 0 \\ 0 & 0 \end{bmatrix}$$

- Choose 2 ^ line pairs (L_a, m_a) and (L_b, m_b)

- These satisfy $L^T C^*_{\infty}{}' m = 0$ or:
- (note x_3 term is ignored!)

$$\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} s1 & s2 & 0 \\ s2 & s3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

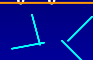
- ‘flatten’ to one equation:

$$\begin{bmatrix} l_1 m_1 & l_2 m_1 + l_1 m_2 & l_2 m_2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0$$

Undoing H: Metric Rect. 1a

Method 1a: (pg 36)   Assume v=0, solve for K

$$C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix} \rightarrow \begin{bmatrix} s1 & s2 & 0 \\ s2 & s3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



- ‘Stack’ to combine both line pairs:

$$\begin{bmatrix} l_1 m_1 & l_2 m_1 + l_1 m_2 & l_2 m_2 \\ l_1 m_1 & l_2 m_1 + l_1 m_2 & l_2 m_2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0$$

- Solve **s** using SVD: ‘input null space’ (Ax=0)
- Extract $\mathbf{H}_A$ using SVD:
  recall $C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T$, it is symmetric…

Undoing H: Metric Rect. 1b

Method 1b: (not in book)   Assume K=I, solve for v

$$C^*_{\infty}{}' = H\, C^*_{\infty}\, H^T = \begin{bmatrix} KK^T & Kv \\ v^TK & 0 \end{bmatrix} \rightarrow \begin{bmatrix} I & v \\ v^T & 0 \end{bmatrix}$$

- Choose 2 ^ line pairs (L_a, m_a) and (L_b, m_b)

- These satisfy $L^T C^*_{\infty}{}' m = 0$ or:

$$\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & v1 \\ 0 & 1 & v2 \\ v1 & v2 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = 0$$

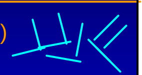
- ‘flatten’ to one equation (messy), stack, solve for $\mathbf{v_1, v_2}$
- Extract $\mathbf{H}_p$ from $C^*_{\infty}{}'$ using SVD

Undoing H: Metric Rect. 2

Method 2: Rearrange, or 'flatten' $C^*_{¥}{}'$

$$C^*_{¥}{}' = H\, C^*_{¥}\, H^T = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

- Choose **5** ^ line pairs $(L_1, m_1) \ldots (L_5, m_5)$



- These satisfy $L^T C^*_{¥}{}'\, m = 0$ or:

$$\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \mathbf{0}$$

- 'flatten' to one equation:

$$\begin{bmatrix} l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \mathbf{0}$$

Undoing H: Metric Rect. 2

Method 2: Rearrange, or 'flatten' $C^*_{¥}{}'$

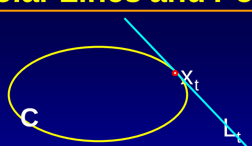
$$C^*_{¥}{}' = H\, C^*_{¥}\, H^T = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

- stack for all 5 line pairs

$$\begin{bmatrix} l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \\ l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \\ l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \\ l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \\ l_1 m_1 & (l_2 m_1 + l_1 m_2)/2 & l_2 m_2 & (l_3 m_1 + l_1 m_3)/2 & (l_2 m_3 + l_1 m_2)/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \mathbf{0}$$

- Solve for a,b,c,d,e,f with SVD (null space; Ax=0)
- Extract H_p from $C^*_{¥}{}'$ using SVD

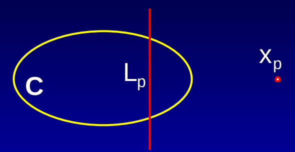
Polar Lines and Pole Points



- Line Conic **C**'s tangent line L_t at point x_t by:

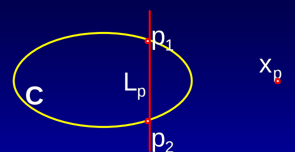
  $C\, x_t = L_t$ (given x_t is on the conic: $x_t^T C\, x_t = 0$)

Polar Lines and Pole Points



- Line Conic **C**'s tangent line $\mathbf{L}_t$ at point x_t by:
  $\mathbf{C\,x}_t = \mathbf{L}_t$ (given $\mathbf{x}_t$ is on the conic: $\mathbf{x}_t^T\,\mathbf{C}\,\mathbf{x}_t\mathbf{=0}$)
- But if **x** is NOT on the conic? try $\mathbf{Cx}_p = \mathbf{L}_p$

Polar Lines and Pole Points



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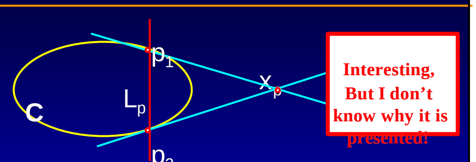
- But if **x** is **NOT** on the conic? try $\mathbf{Cx}_p = \mathbf{L}_p$
- 'Polar line' L_p = conic at p_1, p_2 (find them?ugly!)

Polar Lines and Pole Points



- Line Conic **C**'s tangent line $\mathbf{L}_t$ at point x_t by:
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- But if **x** is **NOT** on the conic? try $\mathbf{Cx}_p = \mathbf{L}_p$
- 'Polar line' L_p = conic at p_1, p_2 (to find them?ugly!)
- p_1, p_2 tangent lines meet at 'Pole point' x_p

SVDs and Conics

- Conics (both C and C*) are symmetric;
- SVD of any symmetric A is also symmetric:
  $SVD(A) = USU^T$
- All conic's singular values s_i =0,1, or –1.
- Singular values classify conic type: (pg 40)

S_i values	Equation	Type
(1, 1, 1)	$x^2 + y^2 + w^2 = 0$	imaginary-only
(1, 1,-1)	$x^2 + y^2 - w^2 = 0$	circle
(1, 1, 0)	$x^2 + y^2 = 0$	single real point (0,0,1)
(1,-1, 0)	$x^2 - y^2 = 0$	2 lines: x+/- y
(1, 0, 0)	$x^2 = 0$	2 co-located lines: x=0

Eigen-values,-vectors, Fixed pt & line

- Formalizes 'invariant' notion:
  - if x is 'fixed' for H, then Hx only scales x
    H x = λ **x** (λ is a constant scale factor)
  - **x** is an 'eigenvector', λ is its 'eigenvalue'
  - again, SVD helps you find them.
- Elaborate topic (but not hard). Skip for now.

- NEXT CLASS:
  - Will post Homework 2, update schedule
  - Will begin Chapter 2, '3D Projective Geometry'

END