## CS 395/495-26: Spring 2002

# IBMR: Week 4 A

# Chapter 2: 3D Projective Geometry

Jack Tumblin
jet@cs.northwestern.edu

## 3D Homogeneous Coordinates

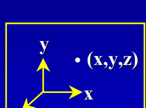
Extend Projective space from 2D to 3D:

- **$P^2$**:
  - From 2D 'world space' (x,y) make
  - 2D homogeneous coordinates $(x_1,x_2,x_3)$.
  - 2D projective 'image space' $(x',y') = (x_1/x_3,\ x_2/x_3)$
- **$P^3$**:
  - From 3D 'world space' (x,y,z) make
  - 3D homogeneous coordinates $(x_1,x_2,x_3,x_4)$.
  - 3D projective 'image space' $(x',y',z') = (x_1/x_4,\ x_2/x_4,\ x_3/x_4)$

## 3D Homogeneous Coordinates

- Unifies points and planes
- (but **lines** are messy)
- Puts perspective projection into matrix form
- No divide-by-zero, points at infinity defined…

in **$R^3$**, write point **x** as $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$



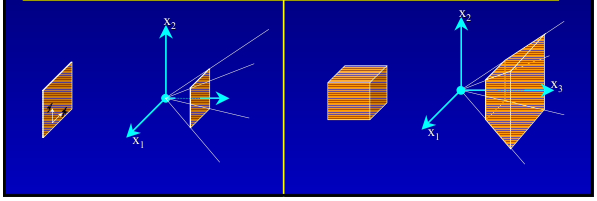

But in **$P^3$**, write same point **x** as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where:

$x = x_1 / x_4,$
$y = x_2 / x_4,$
$z = x_3 / x_4,$
$x_3$ = anything non-zero!
(but usually defaults to 1)

## A Very Common Mistaek:

- $P^2$ homog. coords: 2D projective map
  (2D point→3D ray is correct)
- $P^3$ homog. coords. 3D projective map
  ( But 3D point→3D ray' is WRONG!)

(x,y) ⟶ $(x_1, x_2, x_3)$ | (x,y,z) ⟶ $(x_1, x_2, x_3, x_4)$




## A Very Common Mistaek

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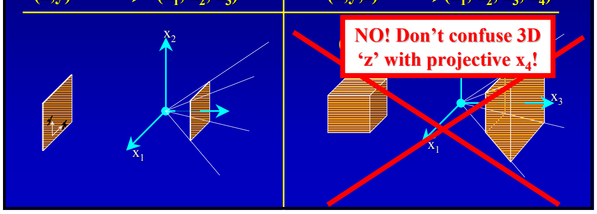
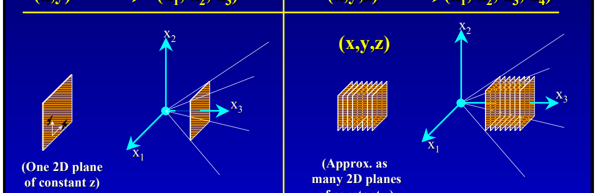



## 3D Homogeneous Coordinates

- $P^2$ homog. coords:  2D projective map
  (2D point→3D ray)
- $P^3$ homog. coords. 3D projective map
  3D point→4D 'ray':
  --an impossible-to-draw **plane** in $R^4$
  --its 3D part is a ray through origin

(x,y) ⟶ $(x_1, x_2, x_3)$ | (x,y,z) ⟶ $(x_1, x_2, x_3, x_4)$



## $P^3$: Point ←→ Plane duality

Recall Plane Equations in 3D: $ax + by + cz + d = 0$

- Unaffected by scale k, with $kax + kby + by + kd = 0$
- Normal vector: (a,b,c) = $\hat{n}$
- Min. Distance from origin: d

Write in 3D homog. coordinates:

- Point **x** and Plane **π** are duals **(lines are not!)**

$ax + by + cz + d = 0$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0} \qquad \mathbf{x}^T \cdot \boldsymbol{\pi} = \mathbf{0}$$

## Points ←→ Planes in $P^3$

- **Join 3 points $P_1$, $P_2$, $P_3$ to find a plane $\pi$:** $\begin{bmatrix} \mathbf{P}^T_1 \\ \mathbf{P}^T_2 \\ \mathbf{P}^T_3 \end{bmatrix} \pi = \mathbf{0}$
  - Stack points:
  - Find null space
  - (Rank 2? collinear points!) (SVD is a good idea…)

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \mathbf{0}$$

- **OR**: use 3D cross products (drop $x_4$):

$$\pi = \begin{bmatrix} (\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3) \\ \vdots \\ \hline -\mathbf{p}_2^T \cdot (\mathbf{p}_2 \times \mathbf{p}_3) \end{bmatrix}$$

(constructs plane normal & length d by cross-products)

## Points ←→ Planes in $P^3$

- To parameterize 3D space by plane $\pi$:

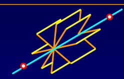
  (e.g. find u,v coordinates for all of 3D)
  - Recall $\pi$ is equiv. to plane's normal vector
  - Find 3 ⊥ vectors (*e.g.* null space of [**0** ⋮ **0** ⋮ **0** ⋮ $\pi$] )
  - assemble them as columns of 4x3 **M** vector
  - Apply to <u>any</u> point **p**:

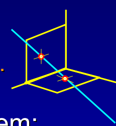
**M p = uv**

  - Finds a 2D coords in $\mathbf{P}^2$ plane (of $\pi$): uv = $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

## Lines in $P^3$: Awkward

- Geometrically:
  - Intersection of 2 (or more) planes
  - Line defines an axis or 'pencil' of planes
  - Linear combo of 2 points $(p_1 + A(p_2 - p_1))$
  - Have **4** DOF in $P^3$; 4-vector won't do.

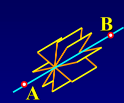




- Symbolically: three ways to write them:
  - Span or Null Space
  - Plűcker Matrix
  - Plűcker Line Coordinates

## $P^3$ Lines 1a: (Plane) Span

- Recall that a point $\mathbf{x}$ is on a plane $\pi$ if
  $\mathbf{x}^T \cdot \pi = 0$
- 2 given points **A, B** intersect with a 'pencil' of planes on a line



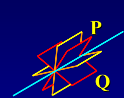

- Write line as that intersection: stack $\mathbf{A}^T$, $\mathbf{B}^T$ to make 2x4 matrix **W:**

$$\mathbf{W} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

- **If line W contains the plane $\pi$, then** $\boxed{\mathbf{W}\,\pi = 0}$

## $P^3$ Lines 1b: (Point) Span

- Recall that a point $\mathbf{x}$ is on a plane $\pi$ if
  $\mathbf{x}^T \cdot \pi = 0$
- 2 given planes **P,Q** intersect at a 'pencil' of points on a line



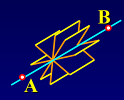

- Write line as that intersection: stack $\mathbf{P}^T$, $\mathbf{Q}^T$ to make 2x4 matrix **W*:**

$$\mathbf{W}^* = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

- **If line W* contains the point x, then** $\boxed{\mathbf{W}^*\,\mathbf{x} = 0}$

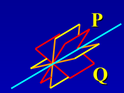
## $P^3$ Lines 1:Spans

- (Point) Span W
  - (Found from points **A,B**)
  - Used to test plane $\pi$: $\boxed{W \pi = 0}$




$$\boxed{W^T W^* = W^* W^T = 0_{2x2} \text{ (the 2x2 null matrix)}}$$

- (Plane) Span W*
  - (Found from planes **P,Q**)
  - Used to test point $\mathbf{x}$: $\boxed{W^* \mathbf{x} = 0}$




## $P^3$ Lines 1:Spans

- **Join Line W and Point p → Plane $\pi$**

  $W \pi = 0$ iff plane $\pi$ holds line **W**, and
  $p^T \pi = 0$ iff plane $\pi$ holds point **p**; stack …

  Let $M = \begin{bmatrix} W \\ -- \\ p^T \end{bmatrix}$ ; solve for $\pi$ in $\boxed{M \pi = 0}$

- **Join Line W* and Plane $\pi$ → Point p**

  $W^* \mathbf{p} = 0$ iff line **W*** holds point **p**
  $\pi \mathbf{p} = 0$ iff plane $\pi$ holds point **p**; stack…

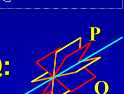
  Let $M^* = \begin{bmatrix} W^* \\ --- \\ \pi \end{bmatrix}$ ; solve for $\mathbf{p}$ in $\boxed{M^* \mathbf{p} = 0}$

## $P^3$ Lines 2: Plűcker Matrices

Line == A 4x4 symmetric matrix, rank 2, 4DOF


- Line **L** through known points **A, B:**

  $$L = AB^T - BA^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$




- Line **L*** through known planes **P,Q:**

  $L^* = PQ^T - QP^T$

## $P^3$ Lines 2: Plűcker Matrices

Line == A 4x4 symmetric matrix, rank 2, 4DOF

- Line **L** through **A, B** pts: **$L = A \cdot B^T - B \cdot A^T$**
- Line **L*** through **P,Q** planes: **$L^* = P \cdot Q^T - Q P^T$**

- **L → L*** convert? Note that:

$$\begin{bmatrix} \cdot & l_{12} & l_{13} & l_{14} \\ \cdot & \cdot & l_{23} & l_{24} \\ \cdot & \cdot & \cdot & l_{34} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \text{ or } \begin{bmatrix} \cdot & l_{12} & l_{13} & l_{14} \\ \cdot & \cdot & l_{23} & \cdot \\ \cdot & \cdot & \cdot & l_{34} \\ \cdot & l_{42} & \cdot & \cdot \end{bmatrix}$$

(to avoid minus signs)

  - Skew-symmetric;
  - L's has 6 params:
  - Note det|L|=0 is written $l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$
  - **Simple!** Replace $l_{ij}$ with $l_{mn}$ so that {i,j,m,n} = {1,2,3,4}
    Examples: $l_{12} \rightarrow l_{34}$, or $l_{42} \rightarrow l_{13}$ etc.

## $P^3$ Lines 2: Plűcker Matrices

- **Join Line L* and Point p → Plane π**

  $L^* p = \pi$

  (if point is **on** the line, then $L^* p = 0$)

**Join Line L and Plane π → Point p**

$L \pi = p$

(if line is **in** the plane, then $L \pi = 0$)

## $P^3$ Lines 3: Plűcker Line Coords

**==Plűcker Matrix has 6 vital elements:**

- Off-diagonals

$$\begin{bmatrix} \cdot & l_{12} & l_{13} & l_{14} \\ \cdot & \cdot & l_{23} & l_{24} \\ \cdot & \cdot & \cdot & l_{34} \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \text{ or } \begin{bmatrix} \cdot & l_{12} & l_{13} & l_{14} \\ \cdot & \cdot & l_{23} & \cdot \\ \cdot & \cdot & \cdot & l_{34} \\ \cdot & l_{42} & \cdot & \cdot \end{bmatrix}$$

(to avoid minus signs)

- Just make them a 6-vector:

$$\mathcal{L} = \begin{bmatrix} l_{12} \\ l_{13} \\ l_{14} \\ l_{23} \\ l_{42} \\ l_{34} \end{bmatrix}$$

and require that det|L| = 0, or:

$$l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$$

## Projective Transformations

- Use H for transforms in $P^3$:
  - Has 15 DOF (4x4 -1)
  - Superset of the $P^2$ **H** matrix:

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \rightarrow \begin{bmatrix} h_{11} & h_{12} & 0 & h_{13} \\ h_{21} & h_{22} & 0 & h_{23} \\ 0 & 0 & 0 & 0 \\ h_{31} & h_{32} & 0 & h_{33} \end{bmatrix}$$

## $P^3$ Transformations

- Transform a point **p** or plane $\pi$ with **H**:

$$p' = H \cdot p \qquad \pi' = H^{-T} \cdot \pi$$

- Lines 1: Transform a span:

$$W' = H \cdot W \qquad W^{*\prime} = H^{-T} \cdot W^*$$

- Lines 2: Transform a Plűcker Matrix:

$$L' = H \cdot L \cdot H^T \qquad L^{*\prime} = H^{-T} L^* \cdot H^{-1}$$

# END