CS 395/495-26: Spring 2002

# IBMR: Week 6A

## Chapter 3: Estimation & Accuracy

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## Reminders

No midterm, no final, but …
Alternating homework / project

- Project 2 Due Thurs May 9
- Homework 1 due Thurs May 16

- Coming Thursday: Project 3

## Direct Linear Tranform (DLT)

- For each point pair $(x_i, x_i')$ Solve $Hx \times x' = 0$
  - Robust! Accepts **any** scale ($x_3=w\neq1$, $x_3'=w'\neq1$), finds **any** **H**
- 'Vectorize'; make 2 rows of **A** per point pair
- Solve $A_i h = 0$ stacked up…

$$\begin{bmatrix} 0 & 0 & 0 & -w'x & -w'y & -w'w & y'x & y'y & y'w \\ w'x & w'y & w'w & 0 & 0 & 0 & -x'x & -x'y & -x'w \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = 0$$

…(or written in book's vector notation (pg 71, 3.1)…)

$$\begin{bmatrix} 0^T & -w'_i \mathbf{x}_i^T & y'_i \mathbf{x}_i^T \\ w'_i \mathbf{x}_i^T & 0^T & -x'_i \mathbf{x}_i^T \\ -y'_i \mathbf{x}_i^T & -x'_i \mathbf{x}_i^T & 0^T \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = 0$$

## Adding More Measurements

If we use >4 point correspondences?

Quick-and-sloppy:

- Adds row pairs to our 8x9 matrix A: **$A \cdot h = 0$**
- Use SVD to find Null space (Always gives an answer!)
- **Result:** 'Least squares' solution
  - minimizes 'algebraic distances' $\epsilon_i^2$ between point pairs.

$$\| \mathbf{A}\,\mathbf{h} \|^2 = \epsilon^2 = \| \text{tall, near-zero vector} \|^2 = \begin{bmatrix} \epsilon_1^2 + \\ \epsilon_2^2 + \\ \epsilon_3^2 + \\ \cdots \end{bmatrix}$$

$\sum_i \epsilon_i^2 =$ where $\epsilon_i^2$ is error for i-th pt. correspondence:

$\epsilon_i^2 = \| Hx_i \times x_i' \|^2 = \| (\text{2 rows of A}) \cdot h \|^2 =$ 'algebraic distance'

## Adding More Measurements

- **2D 'Algebraic Distance' ?** No geometric meaning!
- **2D 'Geometric Distance' $d(a,b)^2$** is better: measurable length in input or output space

  if $\mathbf{a} = (a_1\ a_2\ a_3)$ and $\mathbf{b} = (b_1\ b_2\ b_3)$, then define

$$d(\mathbf{a},\mathbf{b})^2 = \left(\frac{a_1}{a_3} - \frac{b_1}{b_3}\right)^2 + \left(\frac{a_2}{a_3} - \frac{b_2}{b_3}\right)^2$$

**Turns out that:** (for $P^2$)

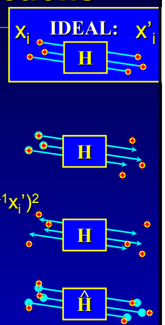
$$d(\mathbf{a},\mathbf{b}) = \frac{d_{algebraic}(\mathbf{a},\mathbf{b})}{a_3 \cdot b_3}$$

## Adding More Measurements

**Overall Strategy:**

- Overconstrain the answer **H**
  - Collect extra measurements (>4 point pairs, etc. …)
  - **expect** errors; use adjustable 'estimates' $\hat{x}$
- Compute a 1st solution (probably by SVD)
- Compute error $d(H\hat{x}, \hat{x}')^2$, and use this to…
- 'Tweak' answer H **and** estimates $\hat{x}$
- Compute new answer
- Stop when error < useful threshold

## Error Measures & Corrections




- One-Sided:
  - Perfect inputs $x_i$, Flawed outputs $x_i'$
  - Find **H** to minimize $\Sigma_i\, d(Hx_i, x_i')^2$
- Two-sided:
  - Flawed input $x_i$, Flawed output $x_i'$
  - Find **H** to minimize $\Sigma_i\, d(Hx_i, x_i')^2 + d(x_i, H^{-1}x_i')^2$
- Reprojection:
  - both input $x_i$ and output $x_i'$ have errors
  - Find 'estimates' of all: $\hat{x}_i$ and $\hat{x}_i'$ and $\hat{H}$
  - Choose for a perfect match: $\mathbf{H}\hat{x}_i = \hat{x}_i'$
  - Minimize estimates .vs. real: $\Sigma_i\, d(\hat{x}_i, x_i)^2 + d(\hat{x}'_i, x_i')^2$

## HOW? Reprojection in $R^4$

- **Rearrange** Each point pair to make a 4-vector $X_i$:
  $(x_i, x_i') \rightarrow \mathbf{X}_i = [x,y,x',y']^T$ (assume w=w'=1)
- 1) Define an $\mathbf{R^4}$ 'measurement space' for all $X_i$
  - (CAREFUL! this is NOT homogeneous P3: it's true 4-D)
  - holds all possible point pairs, from perfect to horrid
- 2) Find the "shape of perfection" for **H,**
  - What $X_i$ points in $R^4$ are an error-free fit to **H?**
  - .e.g. find all $X_i$ points where $\mathbf{H}\, \mathbf{x}_i \times \mathbf{x}_i' = 0$
  - Any 'estimate' $\hat{X}_i$ we choose will be on this shape
- 3) Find nearest estimate $\hat{X}_i$ for measured point $X_i$

## What's the 'Perfect Shape' in $R^4$?

- Recall a point pair $X_i$ sets two rows of **Ah=0**:

$$\begin{bmatrix} 0 & 0 & 0 & -w'x & -w'y & -w'w & y'x & y'y & y'w \\ w'x & w'y & w'w & 0 & 0 & 0 & -x'x & -x'y & -x'w \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = 0$$

- Rewrite one row of **Ah=0** using $X_i$ and get…
  (rather messy) quadric in $R^4$ ('multilinear'; nothing is squared)

$$\mathbf{X}^T Q \cdot \mathbf{X} + \mathbf{X}^{T} \cdot P + C = 0$$

(Q,P,C are vector/matrix constants)

$$\mathbf{X}^T \cdot \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \end{bmatrix} \cdot \mathbf{X} + \mathbf{X}^T \cdot \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} + C = \mathbf{0}$$

- Two rows → 2 quadrics → a 'variety' $\mathcal{V}_H$

## Reprojection: The Big Idea

- Given **H**, find its **$R^4$** 'variety' $\mathcal{V}_H$ (its 2 quadrics)
- For each measured $X_i$, find nearest estimate $\hat{X}_i$ on $\mathcal{V}_H$ by 'projection'
- **VERY** nice result (robust!):
  - Minimizes input, output geometric error
  - Is also the 'ML' (maximum likelihood) estimate
  - Better than DLT: invariant to origin, scaling, similarity,

**BUT**

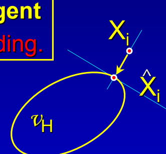
in $R^4$, finding 'nearest estimate' $d_\perp(X_i, \mathcal{V}_H)$ is a mess!

Look at simplified case…

## Try $R^2$ (not $R^4$) first…

Find $d_\perp(X_i, \mathcal{V}_H)$ for $R^2$ to an '$x^2$ family' curve:

- Instead of (x,y)→(x',y') correspondence, (x)→(y)
- Instead of $X_i$ = (x,y,x',y'), use $\mathbf{X}_i$ = (x,y)
- Replace 4D 'Variety' $\mathcal{V}_H$ with 2D conic curve C
- Estimate $\hat{X}_i$ is on conic: $\hat{X}_i^T C \hat{X}_i = 0$
- **Line from $X_i$ to $\hat{X}_i$ is $\perp$ to C tangent**
- **Finding $\hat{X}_i$ from $X_i$, C is root-finding.**




## $R^4$ Solutions: Sampson Error

Find closest point on variety $\mathcal{V}_H$ ?

- No analytical solution (?find is quartic roots!?)
- Linear approx. is MUCH easier: 'Sampson'
- Repeat, improve until estimate is good enough.

- Recall that one $X_i$ point sets 2 rows of Ah=0,
  - But nonzero if H, Xi if don't match: $A_i h = \epsilon_i$

$$\begin{bmatrix} 0 & 0 & 0 & -w'x & -w'y & -w'w & y'x & y'y & y'w \\ w'x & w'y & w'w & 0 & 0 & 0 & -x'x & -x'y & -x'w \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

## $R^4$ Solutions: Sampson Error

$$\begin{bmatrix} 0 & 0 & 0 & -w'x & -w'y & -w'w & y'x & y'y & y'w \\ w'x & w'y & w'w & 0 & 0 & 0 & -x'x & -x'y & -x'w \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

- Rearrange $A_i h = \epsilon_i$ as quadrics of $X_i$
  - Tricky! stacked quadrics compute error:

$$\mathbf{X}^T \mathbf{Q}_1 \cdot \mathbf{X} + \mathbf{X}^T \cdot \mathbf{P}_1 + C_1 = [\epsilon_1]$$
$$\mathbf{X}^T \mathbf{Q}_2 \cdot \mathbf{X} + \mathbf{X}^T \cdot \mathbf{P}_2 + C_2 = [\epsilon_2]$$

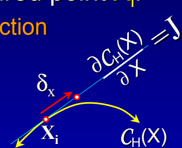
(Q,P,C are vector/matrix constants)

  - Book renames this stack as: $\mathcal{C}_H(X) = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$

## $R^4$ Solutions: Sampson Error

The $\mathcal{C}_H(X) = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$ ==error is a vector **polynom.**

- Do Taylor Series at measured point $X_i$:
  - Follow -(error gradient) direction
  - distance*grad = error




- Estimate $\hat{X}_i = X_i + \delta_x$
- Error vector $\boxed{\delta_x = -J^T(JJ^T)^{-1}\epsilon}$ (J = partial deriv. matrix)

# END