# CS395/495: IBMR Take-Home Midterm Exam

Assigned: 3:30pm,Thurs May 1, 2003
Due: 3:30pm,Thurs May 8, 2003

You may turn in your exam work either on paper brought to class, or by e-mail to me (jet (at) cs (dot) northwestern.edu). If you submit your work on paper, be sure to number the pages and put your name on each page. Please do your own work—do not discuss the problems or your solutions with other students until after the due date.

## Projective 2-D Exercises:

1) What 3-vector describes the line in $P^2$ that intersects the x axis at (xp,0) and the y axis at (0,yp) ?

   **Cross-product of two points in P2 produces the line that passes through both of them. Define p1 = (xp,0,1) and p2 = (0,yp,1), then find line L = p1 × p2 =**

   **line L = $[-yp, -xp, xp*yp]^T$ (or L*d where d is any nonzero scalar value).**

2) What point is at the intersection of the lines [a b c]^T and [d e f]^T?

   **Cross-product of two lines in P2 produces the point at their intersection:**

   **point P = $[(bf-ce), (cd-af), (ae-bd)]^T$**

3) Show mathematically that two parallel lines in P2 space really do intersect at infinity, and their intersection point is part of the $L_\infty$ line.

   **If line L1 is [a b c]T and line L2 is parallel to it, then L2 = [a b d] where d!=c. You can find their point of intersection using the cross-product:**
   **L1 × L2 = $[b(d-c), a(c-d), 0]^T$ . Any point in P2 where x3=0 is infinitely far from the origin; it is an 'ideal point', and all ideal points $p_I$ $[x,y,0]^T$ are points on the line at infinity $L_\infty = [0,0,1]^T$ because $L_\infty^T \cdot p_i = 0$.**

4) Vanishing points and panoramas: Suppose we visit a farmer's field in Kansas to take pictures, where the ground is flat and level for many miles. The farmer's field plowed in long, straight, parallel furrows that reach towards the horizon. However, one set of railroad tracks cut across the farmer's field. The tracks are perfectly straight and level for many miles. The furrows are not parallel to the railroad tracks. Now suppose we stand in one place, aim the camera north, take one picture, turn about 20 degrees eastward, take another picture, turn another 20 degrees eastward, and take a third picture. There is considerable overlap in each picture, because the camera's horizontal field of view is 38 degrees. In each picture, we can see both furrows and the railroad tracks, and it is easy to define parallel line pairs for each of them. We took the photographs in springtime, and the machine-planted crop is just beginning to sprout, so that each furrow has a small green spot spaced equally along each furrow.

   On the second digital photograph, we define two pairs of parallel lines, one pair that matches the two railroad tracks, given by **La**=(La1,La2,La3), **Lb** (Lb1,Lb2,Lb3), and the other pair is aligned with two furrows in the farmer's field, given by **Lc** = (Lc1,Lc2,Lc3) and **Ld** = (Ld1,Ld2,Ld3).

a) Find the transformation matrix H needed that will transform these into parallel lines aligned with the y axis (x2/x3) in P2. After this transformation, the image would appear to be an aerial view: all furrows will be parallel to each other, and the railroad tracks will be parallel.

**There are several ways to solve this problem. I used the vanishing lines method for projective rectification, then rotation to align with the y axis. The problem statement also doesn't specify which line pair to align with the y axis; you may use either one, but I chose the second one.**

**A 4-point correspondence solution is possible, but will force you to make two assumptions about the spacing between the sprouting plants. 1) aspect ratio: is the spacing between furrows the same as the spacing between plants on one furrow? 2) angle: what is the angle between the furrows and a line connecting adjacent plants on different furrows?**

**My solution:**

**--The image plane contains a horizon line Lh (it does not have to be visible in the photograph); find two points (p1, p2) on the horizon line by using cross-products:**

**$p1 = La \times Lb$, $p2 = Lc \times Ld$**

**--Find the horizon line Lh by the cross product of these two points:**

**$Lh = p1 \times p2$ [Lh1, Lh2, Lh3]**

**--Then the projective transformation Hp that converts the line at infinity $L_\infty$ to the horizon line Lh must be:** $Hp^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Lh1 & Lh2 & Lh3 \end{bmatrix}$ **(I use the * to denote this transform applies to lines, not points) (see page 29)**

**--We want to undo the effects of Hp*, so we use its inverse, $(Hp^*)^{-1}$, but we also want to apply it to points in the image, so we take its inverse transpose (because $Hp^* = Hp^{-T}$; see pg. 15) and get**

**$H1 = Hp^{*T}$**

**--If we apply H1 to every point in the image we will remove the projective transformation, but we still have an unknown rotation. We want the transformed line $Lt = (Hp^*)^{-1} \cdot La$ (or Lb,Lc, or Ld—your choice) aligned with the y-axis line Ly=(1,0,0) We find rotation amount by equation 1.20 (pg. 33):** **$\cos\theta = (Ly1*Lt1 + Ly2*Lt2) / \text{sqrt}((Ly1^2+Ly2^2)(Lt1^2+Lt2^2)) = C$**

**if $C = \cos -\theta$, then let $S = \sin -\theta$, and we can define a rotation-removing matrix H2 as:**

$$H2 = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**--Thus to make the desired view, apply the matrix $H = H2 \cdot H1$ to every point in the image.**

b) Explain how you would find transformation matrices H1 and H3 needed to create a 3-panel panorama. In the panorama, image 2 is not transformed at all—it is centered on the x3 axis (as it would be in your ProjectA and Project B software), and images 1 and 3 are positioned to the left and right of it by applying transformations H1 and H3. For an illustration, see page 196 (but you can do this in P2 only—there is no need for the P3-based method described there). Explain how you would find H1 and H3 in sufficient detail to implement it in your ProjectA or B code.

**For H1: in the overlapped regions of images 1 and 2, find at least 4 corresponding green spots caused by selected sprouting plants that are visible in both images. You may also use endpoints of railroad cross-ties if these are visible in the overlapped region. Be sure to avoid green spots that are co-linear in the images; you can do this by ensuring that no furrow contains more than two chosen points, etc. Use the left-side image (Image 1) as 'input' or 'world' space, and define its points as x1,x2,x3,...; use the center image (Image 2) as 'output' space, and define its corresponding points as x1', x2', x3'..., and then use Direct Linear Transformation to find the homography H1 that transforms points x→x'. Apply H1 to all points in the left-side image,**

**(assume x3=1 before transformation) and use alpha-blending to combine the left and center image into one planar image. Use the same procedure to include the right-side image (image 3).**

# Conics in $P^2$:

5) Suppose you have photograph of a planar table-top, photographed from an unknown position. Within that image you have measured N line pairs that are linearly independent (e.g. not redundant) and named (L1a, L1b), (L2a, L2b), ...(LNa, LNb).

   a) Line pairs 1-5 are known to form 90-degree angles. Find the 3x3 $C^*_\infty$' matrix for this photograph.

   **Recall that if lines L1 and L2 are perpendicular, then $La^T \cdot C^*_\infty \cdot Lb = 0$, and if we apply a transformation H* that converts La → La', Lb→Lb', and $C^*_\infty$→ $C^*_\infty$', the property still holds: $La'^T \cdot C^{*\prime}_\infty \cdot Lb' = 0$. Five 5 linearly-independent pairs of (La', Lb') are sufficient to solve for $C^{*\prime}_\infty$. Written out element-by-element, let:**

   $$La'^T \cdot C^{*\prime}_\infty \cdot Lb' = [a1\ a2\ a3] \begin{bmatrix} c1 & c4 & c5 \\ c4 & c2 & c6 \\ c5 & c6 & c3 \end{bmatrix} \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix} = 0 =$$

   $$(a1 \cdot b1)c1 + (a2 \cdot b2)c2 + (a3 \cdot b3)c3 + (a1 \cdot b2 + a2 \cdot b1)c4 + (a1 \cdot b3 + a3 \cdot b1)c5 + (a2 \cdot b3 + a3 \cdot b2)c6 = 0$$

   **--Collect the known values (in parintheses) into a row vector $A_i$, and the unknown values into a column vector C; then $A_i \cdot C = 0$. Write a different $A_i$ for each of the 5 line pairs (L1a, L1b) ... (L5a,L5b), and stack them to form a 5x6 matrix A. Use Singular Value Decomposition (SVD) to solve for C in the null-space problem AC=0, and C vector elements provide each element of matrix $C^*_\infty$'.**

   b) In a photograph taken from a different position, we measured the angle between lines L7a and L7b as 42 degrees. If the $C^*_\infty$' matrix for this image is:

   | | | |
   |---|---|---|
   | 5.0 | 2.5 | 17.5 |
   | 2.5 | 1.25 | 8.75 |
   | 17.5 | 8.75 | 61.25 |

   then what is the angle between these lines if measured in the plane of the tabletop? (in degrees, not radians).

   **This was a trickier problem than I intended. First, define a rectified image of the tabletop (e.g. camera is directly above the tabletop, and distances and angles measured in pixels are proportional to distances and angles measured in meters and degrees on the tabletop). I will call this rectified version 'image 1, and refer to the new image described in b) as image 2. In image 2 we have two lines separated by 42 degrees; I will call them L' and M'. Our goal is to find the angle in image 1 between the corresponding lines L and M.**
   **--In image 2, the 42 degree angle can be written in more general terms, using equation 1.21 (pg34):**

   $$\cos 42 = L'^T \cdot C^*_\infty \cdot M / \mathrm{sqrt}((L'^T \cdot C^*_\infty \cdot L')(M'^T \cdot C^*_\infty \cdot M'))$$

   **Because we measure the 42 degree angle in the plane of image 2, we use $C^*_\infty$ here and not $C^*_\infty$'; it is just a fancy way of writing eqn 1.20(pg.33). But remember that lines L' and M' in image 2 are also transformed versions of lines L,M in image 1. If we made a similar angle measurement in image 1 using lines L,M we would find the unknown angle θ:**

   $$\cos \theta = L^T \cdot C^*_\infty \cdot M / \mathrm{sqrt}((L^T \cdot C^*_\infty \cdot L)(M^T \cdot C^*_\infty \cdot M))$$

   **Again we're using $C^*_\infty$, because we're measuring angles in the plane of image 1. But now suppose we transform L, M, and the conic $C^*_\infty$ from image 1 to image 2, producing L', M', and $C^*_\infty$'. As eqn 1.21 shows, we can still compute the same unknown angle θ using these**

**transformed versions.**

**To solve for $\theta$, we need to define two lines in image 2 separated by 42 degrees. I defined line L' as the x-axis, or $L' = [1\ 0\ 0]^T$, and defined M' as a line that intersects the x axis at (–cos 42,0) and the y axis at (0,sin 42). For simplicity, write C = cos 42deg., S = sin 42deg., and then use results of problem 1 to write line $M' = [-S,C,-CS]^T$. Note that these image 2 lines are the 'transformed' versions of (unknown) lines L and M in image 1.**

**Next, apply eqn. 1.21 to transformed values L', M', and $C^*_\infty{}'$. Because we are GIVEN $C^*_\infty{}'$, solving for cos $\theta$ is straightforward; using the L', M' I defined, I get:**

**Note that the $C^*_\infty{}'$ is degenerate, with rank of 1 instead of 2 as is usually the case; it corresponds to a homography H that turns an image on-edge, so that the only possible answer is that the angle $\theta$ value MUST be 0 or 180degrees. (For example if I change the value of element 3,3 from 61.25 to 150, then I get $\theta$ = -45.79 degrees).**

**If I had chosen a more conventional matrix with rank=2 instead, then different students could get different answers yet still be correct. Other choices of L', M' may produce a different result for $\theta$, because the (unknown) transformation that converts image 1 into image 2 may include projective terms that distort angles nonuniformly across the image plane.**

6) a) Write the point-conic matrix C for a circle of radius r centered at (x,y) location (a,b).

$$C = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & (a^2+b^2-r^2) \end{bmatrix}$$ **(see page 9)**

b) Write the point-conic matrix Ch for a hyperbola that intersects the x axis at (-1,0) and (1,0).

**In R2, the family of all such hyperbolae satisfy $x^2 - By^2 - 1 = 0$ where B>0. Substitute; x→x1/x3, y→x2/x3, and then multiply both sides by $x3^2$:**

**$x1^2 - B\,x2^2 - x3^2 = 0$. Writing in matrix form,**

$$Ch = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -B & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ **where B>0**

c) Find the homography H that converts this circle C to a hyperbola Ch.

**Again, MUCH tricker than I'd intended—I don't have an analytic solution! As on pg. 15, result 1.13, a transformed point conic C→C' is given by $C' = H^{-T}CH^{-1}$. Thus we must solve for H in the expression $Ch = H^{-T}CH^{-1}$. Post-multiply both sides by H, pre-multiply both sides by $H^T$, and we have $H^T\,Ch\,H = C$. But Ch is very simple—it is almost an identity matrix. In fact, we can define an H1 matrix that turns it into an identity matrix:**

$$H1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i/\text{sqrt}(B) & 0 \\ 0 & 0 & i \end{bmatrix}$$ **where i = sqrt(-1). Note that $H1^T \cdot Ch \cdot H1 = I$**

**If we define the unknown 3x3 matrix H as H1·H2, we can write**

**$(H1 \cdot H2)^T\,Ch\,H1 \cdot H2 = C$**

**$H2^T\,(H1^T\,Ch\,H1)\,H2 = C$**

**$H2^T\,H2 = C$**

**Now you can try do a singular-value decomposition on C: $SVD(C) = USV^T = C$. We know C is square, so that U and V are both square as well. Because C is symmetric, then U=V. The S matrix is diagonal, so we can take the square-root of each element to form the matrix Sq; note that $Sq = Sq^T$, so we can write $S = Sq\,Sq^T$. Combining these ideas,**

**$H2^T\,H2 = C = USV^T = U\,Sq\,Sq^T\,U^T = (U\,Sq)(U\,Sq)^T$**

**Thus one solution may be H = H1 H2 = H1 (U Sq)$^T$. Any serious, credible attempt at this problem will get full credit; any better solution will get extra credit.**

# Singular Value Decomposition:

7) $P^2$ Conic Correspondence: Describe a method for finding a homography between an input image and an output by matching one or more conic features in the images, such as the circles formed by two wheel rims in the side view of a car or truck, or the image of several CDs placed randomly on a tabletop. Do this in two steps:

a) Describe how you would convert points or lines positioned on a conic curve in the image into the C matrix that describes the conic. (Be specific: your explanation must be sufficiently detailed for someone to implement it a matrix class that includes an SVD-finding function).

**There are several solutions. Given measured lines L1, L2, L3, ...LN that are tangent to a line conic C* at measured points P1, P2, P3,...PN or a point conic C, we know that for every point P, every line L, or every point/tangent line pair P,L:**

**a) C* L = P,**
**b) CP = L,**
**c) $L^T$C*L=0, and**
**d) $P^T$CP=0**

**No matter which equation we choose (a,b,c or d), we can rearrange the knowns and unknowns in a null-space problem to solve by SVD. The unknowns are always the elements of the C or C* matrix, and we can always arrange them as a column vector CC. The knowns for the j-th line or point can always be written as a row vector Aj (or a stacked set of row vectors, as in DLT), so that a),b),c), or d) is written Ai CC = 0.**

**For a) or b), we can write either a 'naïve solution' (for a) use C* L –P=0, or for b), use CP – L =0) or we can make a DLT-like solution that uses the cross-product identity P×P=0. Just as the DLT used Hx × x' = 0, we can also use the cross product: for a) use C*L × P = 0, or for b) use CP × L = 0. Note that c) and d) are already null-space problems; we simply have to rearrange them so that the unknowns and knowns are grouped together.**

**Just as with the DLT, we 'stack' Ai row vectors together, one row (or set of rows) for each line, point, or line/point pair, to form a matrix of knowns 'A'. We take the SVD, find the singular value that is zero, and find the corresponding column of V : its contents is the CC vector we seek.**

**Finally, we rearrange the CC vector into the 3x3 matrix form used by conics.**

b) Describe how you would find a good estimate of homography H from one or more conic pairs (C, C'). It is not necessary to use the $C^*_\infty$' in your solution.

**Several workable solutions exist. One simple way is to use conics to supply accurate point or line estimates for 4-point correspondence/DLT solutions. For example, estimate many points along the boundaries of a known conic curve in the image (e.g. points along the edges of the 'Target' sign used in class notes); these will have noise, but we can use a DLT-like process to fit find the conic parameters (a,b,c,d,e,f) that best fit these points in the image. Then use the conics to estimate locations, such as the centroid of the conic in the image, and the major and minor axes (if the conic is an ellipsoid). Differences between conic centers also define points and lines that can be used in conventional DLT and vanishing point methods.**

# Estimation/Optimization:

8) Construct a specific example of image rectification where the DLT solution of Chapter 3 (e.g. $Hx \times x' = 0$) works well, but the 'naïve' method (pg. 13,14: $Hx - x' = 0$) works poorly or not at all. Your answer should list at least 6 point pairs (**x**,**x'**). Describe the method you used to find this example; it should (Hint: trial and error isn't a good idea; this question doesn't require you to implement the naïve solution in software or compute it by hand).

   **DLT's primary improvement over naïve method is explicit inclusion of w and w' measurements. These allow the DLT to avoid 'divide-by-zero' problems for transformations where either the input or the output points include values where x3 is zero or nearly zero. For example, a transformation that re-orients points in an image-plane to a nearly edge-on view will have almost all differences between point pairs contained in the w and w' components.**

9) Section 3.4 in the book (pp. 88--93) and the last slide of Lecture 7 describe another subtlety of the DLT method: its precision depends on the placement of the origin. Though will not need to implement 'normalization' in your DLT code to answer this question, I want to be sure you know how to do it by solving this problem. Given these 6 noisy point pairs: (input)(output)

   (100.0, 50.0, -1.0) ( 300.0, 250.0, 2.0)
   (101.0, 50.0, -1.0) (300.6, 251.0, 2.0)
   (101.0, 52.0, -1.0) (301.3, 252.0, 2.0)
   (100.0, 49.0, -1.0) (299.5, 248.0, 2.0)
   (105.0, 60.0, -1.0) (305.4, 259.7, 2.0)

   find the normalizing and de-normalizing matrices T and T', and explain how to compute and apply them to improve the accuracy of our DLT solution. Give sufficient detail in your explanation to allow an easy implementation.

**Box on Page 92 outlines the process:**

**--Find the centroid of the input points x, x': xavg = [101.4, 52.2, -1]$^T$ x'avg = [301.36, 252.14,2]T**

**--Find the average distance from each point to its centroid: xdist = 3.475 x'dist = 3.434**

**--Make a T matrix, applied to all input points as Tx = xhat, that shifts centroid to the origin and scales points so the average distance to origin is sqrt(2);**

**T = Tscale Ttrans =**

$$\begin{bmatrix} 0.407 & 0 & 0 \\ 0 & 0.407 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -101.4 \\ 0 & 1 & -52.2 \\ 0 & 0 & 1 \end{bmatrix}$$

**--Make a T' matrix, applied to all output points a T'x' = x'hat, that does the same thing:**

**T' = T'scale T'trans=**

$$\begin{bmatrix} 0.412 & 0 & 0 \\ 0 & 0.412 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -301.36 \\ 0 & 1 & -252.14 \\ 0 & 0 & 1 \end{bmatrix}$$

# Open-Ended Discussion Questions:

10) Write a two-column list that compares IBMR to computer graphics methods, listing as many advantages and disadvantages as you can find. Your goal is to find many *reasons* as possible, not to fill many pages. Write just enough to convey the idea; most ideas can be written in one line, or just a few.

11) Explain as clearly as you can all the ways that projective transformations in $P^2$ differ from Cartesian transformations in either $R^2$ or $R^3$.