CS 395/495 Image Based Modeling and Rendering   Spring, 2004
Take-Home Midterm Exam   Jack Tumblin

Assigned: 4/29/2004
Due: 5/6/2004

1(10 pts) This line in P2 space, $L = [1\ 2\ 3]^T$ is like any other line—it is infinitely long. Find the 'ideal points' that exist on each 'end' of the line (the two endpoints are infinitely far away), and write them in P2 coordinates. Would you say the 'universe' of P2 is a circular?

We know that the P2 point **x** is on line **L** if and only if $L^T \cdot x = 0$. In P2 space we can define ideal points (points at infinity) as [x,y,0]. Solve for [1,2,3]·[x,y,0] = 0, then the solution is any point that satisfies x= -2y, or in other words, an entire line of ideal points [2,-1,0]*d where d is an arbitrary scalar value. But remember, P2-space is scale-invariant; the P2 point (a,b,c) and point (a,b,c)*d represents the same Cartesian point (a/c, b/c). Thus our 'whole line' of ideal points [2,-1,0]*d all represent exactly the same point in Cartesian space, and thus both ends of line L are the same infinitely distant Cartesian point. Point, as if P2 is a map of the surface of an infinite sphere!

2(10pts) In P2 space, what is the line that passes through each of these point pairs? Express your answer as a 3-element column vector, whose last element (x3) is 1.0.
p0 = [2 1 -3]
p1 = [-4 3 -2]

Use cross-product to find the line that passes through each line: p0 x p1 = L.
|x1 x2 x3|
| 2   1 -3| = x1((1)(-2) – (-3)(3)) + x2((-3)(-4) – (2)(-2)) + x3((2)(3)-(1)(-4))
|-4   3 -2|
                =[-2+9,12+4,6+4] = L = [7,16,10] = [0.7 1.6 1.0]

3(10pts) In P2 space, what is the point found at the intersection of lines L0 and L1? Express your answer as a 3-element column vector, whose last element (x3) is 1.0.
L0 = [2 1 -3]
L1 = [-4 3 -2]

Same numerical answer as question 2! Cross-product of two lines gives the point at their intersection. P = L0 x L1, and thus P = [0.7 1.6 1.0].

Using the points and lines defined in questions 2 and 3, find the answers to these questions mathematically—show your work:

    4a)(5pts) Is point p0 found on line L0? On line L1?

If true, then L0·p0=0 and L1·p0 = 0. Try it:
      L0·p0= 2*2 + 1*1 +(-3)(-3) = 4+1+9=14.   P0 is NOT on line L0.
      L1·p0= (-4)2 + (3)(1) + (-3)(-3) = -8 +3 + 9 = 4. P0 is NOT on line L1.

4b)(5pts) Is point p1 found on line L0? On line L1?

L0·p1= p0·L1 = 4;   P1 is NOT on line L0.
L1·p1=(-4)(-4) + (3)(3) + (-2)(-2) = 16+9+4 =29. P1 is NOT on line L1.

4c)(5pts) How far away from L0 is point p0 measured perpendicular to L0?
Hint: write the equation of a line in Cartesian coordinates $(Ax + By +C=0)$. What is the distance from the origin to the nearest point on that line? What homography (H matrix) will move a point $[x1,x2,x3]$ to the origin $[0,0,1]$?

Vectors neatly define the lines and the planes that are perpendicular to them by dot products. For example, given a vector from the origin $V=[a,b]$, we know that the dot-product of V with itself is V's length squared; call it 'd': $d^2 = a^2+b^2$. Any other vector $p=(x,y)$ whose dot-product with V is ALSO d^2 must be part of the plane perpendicular to V (because its projection onto the V vector is the same as V itself).

Thus $V·p=d^2$, and because the V vector is perpendicular to the line (or plane) we defined, the endpoint of V is the point on the line closest to the origin. Expand the vector equation $V·p=d^2$ and into its parts and we get $ax +by =d^2$, or $ax +by - d^2 = 0$. This should look familiar! We can scale this equation by any value without changing it; suppose we divide by the length of vector V so that V becomes unit-length vector V^:

$(a/d)x + (b/d)y - d = 0$, (or equivalently $V^·p – d = 0$, or $(a/d)^2 + (b/d)^2 = 1$). This says we can take ANY P2 line such as $L0 = [a,b,c]$, divide all its elements by $d=sqrt(a^2+b^2)$, and we know the distance from the origin to the nearest point on the line is (c/d).

Now apply it: given $L0 = [2,1,-3]$, we know that the normalized vector $V^ = [2/d,1/d]$, where $d = sqrt(2^2+1^2) = sqrt(5)$, so we can re-write the line as $L0 = [2,1,-3]/sqrt(5)$. Thus the distance from the line L0 to the origin is $sqrt(5)$. But that wasn't the question! Instead, let's transform the line L0 so that the point p0 is at the origin, and THEN find the distance to the origin:

Point p0 is at $[2,1,-3]$, or $[2/3, 1/3, -1]$ in the Cartesian (x,y) plane. To translate p0 to the origin, we use this H matrix:
[1 0 2/3]
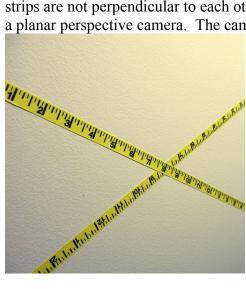[0 1 1/3] $=H_t$.   Then $H_t·p0 = [0,0,1]^T$.
[0 0 1]
To transform L0 in the same way, use $H_t^{-T} L0$ (careful! Don't use the same H matrix on both lines and points in P2 and expect the same results!). Fortunately, the inverse matrix H_t^{-1} is trivial—just translate in the opposite direction (replace 2/3 with -2/3; replace 1/3 with -1/3), and then apply the transpose:
Lnew $= H_t^{-T} L0 =$ [1   0   0] [2] $= [2,1,-4.6667]$
                        [0   1   0] [1]
                        [-2/3 -1/3   1][-3]
For Lnew, we know normalized vector V^ is $[2/d, 1/d]$, and $d=sqrt(5)$, and in Cartesian coordinates the distance from the line Lnew along a perpendicular line to the origin in $(4.6667/d = 4.667/sqrt(5) = 2.087$.

'Project' part of the exam: you will need to use the 'starter code' from the website, and you can find this image as a BMP file "wallTapeRuler.bmp" on the website.

I have a narrow roll of masking tape that is marked in inches like a ruler (from American Science and Surplus). I attached two strips of this tape on a flat portion of a CS department hallway wall, and placed them carefully to be sure each one formed a straight line (note that the tape strips are not perpendicular to each other!) Here is a 512x512 photograph of that wall, taken with a planar perspective camera. The camera is not aligned with the wall.

5a)(15pts) Use the cross-ratio on these two strips of tape in the image to calculate the horizon line L_h for wall, expressed in P2 coordinates for the image. Note that the two strips of tape shown are NOT perpendicular to each other, but they ARE in the same plane. You may make any measurements you wish on this image; assume the image is in the $x_3=-1$ plane, rows and columns of pixels align with x1 and x2 respectively, and the center of the image is at $[0,0,-1]$,

The cross-ratio is an invariant all planar perspective images of points along a line. If we have an image of 4 points that are known to be arranged along a straight line 'world' space, and we measure their positions along the line in the image as (a,b,c,d), then their 'cross ratio' is defined as:

|a-b||c-d|

|a-c||b-d| = cross ratio. If we apply a homography (an 'H' matrix) to the image and measure those points again in the transformed image (e.g. a',b',c',d') then the cross ratio does not change.

We can verify this with the WallTapeRuler.bmp image shown above. I chose 4 convenient points along each ruler strip, and measured their positions (in pixels) in the image using the Windows "paint" utility, which puts the image origin at the upper left corner of the image. For the ruler strip that is lowest on left and highest on right, I defined the ruler marking at:

2" (lower left) as 'world space	a= 0" at the pixel	(107,498) or $t_a = 0$	
10"	b= 4"	(279,352)	$t_b = 225.61$
6"	c= 8"	(400,249)	$t_c = 384.51$
2"(upper right)	d=12"	(486,174)	$t_d = 498.61$ The 't'

measures are distances measured in pixels from point 'a' in the direction of point 'd'.

You can find 't' for all 4 points by defining a 1-D coordinate system with point 'a' as the origin, a unit-length vector P^ pointing from point 'a' to point 'd', and parameter 't' as a multiplier for unit vector P^ added to origin point 'a'. Then the equation of the line is

L(t) = **a** + t·**P^**

P = ((d point) – (a point)) = ((486,174) – (107,498)) = (379,-324),

P^ = P/sqrt(379^2 + 324^2) = P/498.61 = (0.760,-0.650)

So 't' value for point a is t_a=0 and for point d is t_d=498.61.

Now use P^ to measure distance 't' for points b and c along the line in the image; get vector from 'a' to the point b (or c), and find its dot-product with unit vector P^:

(b-a)·P^ = ((279,352) – (107,498))· (0.760,-0.650) = 225.60 = t_b

(c-a)·P^ = ((400,249) – (107,498))· (0.760,-0.650) = 384.51 = t_c

For the ruler measurements, the cross-ratio is:

cross(0,4,8,12) = (4)(4) / (8)(8) = 1/4 = 0.25

For the image measurements,

cross(0,225.61,384.51,498.61) = 0.2452 ---within 0.5%: close enough!

Now; what is the cross-ratio if we make point 'd' infinitely far away on the wall—e.g. where the line on the wall hits the horizon?

cross(0,4,8,∞) = (4)(∞)/(8)(∞) = 4/8 = 0.50.

We can use this to solve for the equivalent point in the photograph: what is the t_d for which the cross-ratio would be 0.50?

The cross ratio R = ((b-a)(d-c))/((c-a)(d-b)); all are known except d:

R(c-a)/(b-a) = (d-c)/(d-b); let A= R(c-a)/(b-a) (a known value) and then write

A(d-b) = d-c; dA-bA=d-c; dA-d = bA-c; d(A-1) = bA-c

d_new = (bA – c) / (A-1)

Using a=t_a, b=t_b, c=t_c, and the desired X=0.5, we get

A = 0.852, d_new = 1300.5 pixels, or about 2.5 image-widths away. To find the pixel address of this 'horizon' point in the image, use **L**(t) = **a** + t·**P^**, or

(x_h, y_h) = (107,498) + 1300.5(0.760,-0.650) = (1095.5, -347.0). But I used windows' 'Paint' program to gather pixel locations, and it puts the origin at the upper left corner of the image. To put the origin at the image center and make y value increase upwards, let xnew = x-256 , ynew = 256-y:

so if x,y origin is at image center, **then 1ˢᵗ horizon point is at pixel (839.5,603.0),** even though the image we have spans only +/- 256 pixels in x,y.

Repeat the process for the ruler strip that is highest on left and lowest on right, but use the parameter name 's' instead of 't':   For my measurements, P^ = (0.893,0.450) and:

		at pixel	and	
1"(upper left)	a = 0"	(24,117)		s_a = 0
5"	b = 4"	(240,225)		s_b = 241.49
9"	c = 8"	(372,291)		s_c = 389.07
3"(lower right)	d =14"	(493,354)		s_d = 526.37

We can again verify the cross-ratio matches for both the ruler measurements and the pixel measurements:

cross(0,4,8,14) = 0.30, and cross(sa,sb,sc,sd) = 0.2981, or <1%;close enough!

If we let the 4th ruler mark go to infinity, the cross ratio is again 0.5. If we use 0.5 and the first three image measurements (sa,sb,sc) to find a new sd, we get:

        A = 0.8056, d_new = 1000.47, so the second 'horizon line' pixel would appear at
        A + P^d_new = (24,117) + 1000.47*(0.893,0.451) = (916.9, 568.2)
        Then if we change coordinates so that x,y origin is at the image center, **then**
        2nd horizon point at (660.9, -312.2).

Finally, we take the cross-product of the two horizon points to make the horizon line:

|x1   x2   x3| =
|839.5 603.0 -1|
|660.9 -312.2 -1|

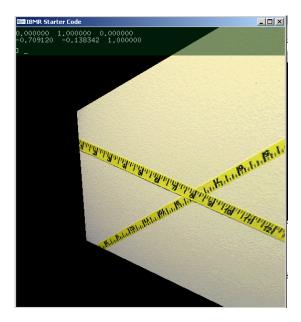$L_h = [-915.3, -178.5, -660,678] = [0.001385, 0.00027, 1.0]$

If you plotted the horizon line L_h on the photo above, you would find it is entirely outside of the photo itself (and not even on the same page of paper). But if you could somehow extend this image, leaving its current set of pixels unchanged, but adding more pixels to extend the width, height, and field-of-view of the image, then you would find that the line L_h location is approximately the end of the hallway in the image—the line where an infinitely large wall vanishes into the distance.

        5b)(5pts) Write the H_p matrix that would 'rectify' this image up to a similarity (you can do this by inspection from your 5a result). Remember, 'up to a similarity' means you're not solving for the entire H matrix; you don't have enough information to solve for rotation and absolute scale.

$H = [$ 1   0   0]   (just put -L_h on bottom row)
          [0   1   0]
        [-0.001385, -0.00027, 1.0]

        5c)(10pts) Using the 'starter code' from the website, transform the images above by the H_p matrix you computed for each of them; make a 'before' and 'after' picture; it should look like the camera moved so that the wall is parallel to the image plane.
        (I used the H matrix above, but scaled the 1st 2 elements of bottom row by 512 to use the same units as the starter code)

The two strips to tape shown in the image above have ¼-inch marks along only one edge (the other is marked in 1-inch marks); use only the ¼-inch-marked edge. Estimate the image coordinates of the location of these 5 points on the image as best you can:

> P0: the point where the two edges cross.
>
> P1 through P4: the points that are a distance of 3 inches of the location P0 as measured by the tape. You know that on the wall (but not in the image) we could use P0 as the center of a circle that passes through and P1 through P4. Call this the 'world-space' circle. In image space, it forms an ellipse.

Measuring using Windows Paint, I get: P0 = (359,284),

P1 = (435,218), P2 = (260,234),

P3 = (261,368), P4 = (431,321);

Averaging gives PC = (346.75, 285.25), and by symmetry we get:

P1' = (PC-P1)+PC   P1' = (258.5,352.5), P2' = (433.5,336.5),

P3' = (432.5,202.5), P4' = (252.5,249.5).

Equation of a circle of radius r centered at x0, y0 is

$$(x - x0)^2 + (y - y0)^2 - r^2 = 0 \text{ or}$$
$$(1)x^2 + (-2x0)x + (1)y^2 + (-2y0)y + (x0^2+y0^2-r^2) = 0$$

so C = [1   0   -x0   ]
       [0   1   -y0   ]
       [-x0   –y0   (x0²+y0²-r²)]

6a)(5pts) List the points you just measured, and then ignore them. First, write the matrix C for a point-conic curve that forms a circle centered at position [x0, y0, 1] with a radius of 3.0. This is just an algebra problem to prepare you for the next steps.

To prepare for the next step, find point PC, the center of the image-space ellipse that passes through points P1,P2,P3,P4 in image space. (e.g. PC = ¼*(P1 + P2 +P3 +P4). Use PC to find 4 more points on the ellipse named P1', P2', P3', P4'. Note that a line through the center of an

ellipse crosses the ellipse at two points, and both of those points are at the same distance from the ellipse center.

        6d)(20pts) Explain how you could find the point-conic matrix C that best fits these 8 points in the least-squares sense by formulating it as a null-space problem; e.g. the elements of the unknown C matrix become a single vector 'c', and the 8 known point locations somehow become a big matrix A, and you find 'c' by solving Ac=0, using singular-value decomposition (SVD). Write A matrix and the c vector. Explain how you would use the U,S,V matrices that the SVD computed for you to find the 'c' vector.

In the image plane where x3=1, any conic (including ellipsoids) is determined by its coefficients A,B,C,D,E,F, and every point (x,y) on the conic satisfies:

     $A x^2 + B xy + C y^2 + D x + E y + F = 0$

Given 8 known points on an ellipse, we can find its (unknown) coefficients by solving Ac=0, where c is a column vector holding the unknown values: $c=[A\ B\ C\ D\ E\ F]^T$, and A is a matrix holding one row of known values for each known point:

$$\begin{array}{llllll}
[x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1] = A \qquad\qquad (\text{point } p1 = x_1,y_1, \text{etc.})\\
[x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1]\\
[x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1]\\
[x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1]\\
[x_{1'}^2 & x_{1'}y_{1'} & y_{1'}^2 & x_{1'} & y_{1'} & 1]\\
[x_{2'}^2 & x_{2'}y_{2'} & y_{2'}^2 & x_{2'} & y_{2'} & 1]\\
[x_{3'}^2 & x_{3'}y_{3'} & y_{3'}^2 & x_{3'} & y_{3'} & 1]\\
[x_{4'}^2 & x_{4'}y_{4'} & y_{4'}^2 & x_{4'} & y_{4'} & 1]
\end{array}$$

To solve Ac=0, find SVD(A) to yield U, S, V matrices. The S matrix will be 6x6, but the i-th row will be zero or very nearly so. Choose the i-th column of the V matrix; this is the value of the c vector. Assemble the components of the c vector to make the 3x3 C matrix.

        6e) (20pts)EXTRA CREDIT: use the 'starter code' on the website, or use MATLAB (both give you an SVD routine) to solve for the C matrix that best fits your points. Plot a few points of the conic defined by C on the image above (better yet, draw the conic using OpenGL in the starter code), to see how well you did.