Code analysis *and* transformation

DFA foundation

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We have seen several examples of DFAs

- Are they correct?
- Are they precise?
- Will they always terminate?
- How long will they take to converge?

Outline

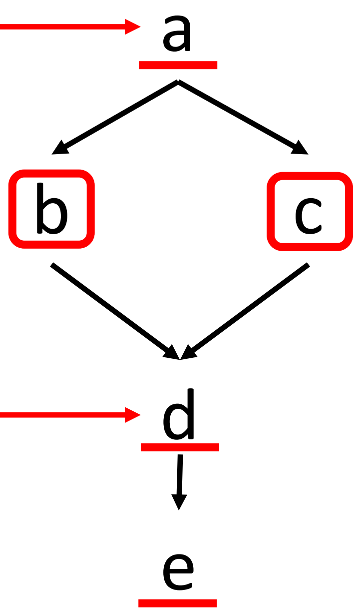
- Lattice and data-flow analysis
- DFA correctness
- DFA precision
- DFA complexity

Understanding DFAs

- We need to understand **all** of them
  - Liveness analysis: is it correct? Precision? Convergence?
  - Reaching definitions: is it correct? Precision? Convergence?
  - …
- **Idea**: create a framework to help reasoning about them
  - Provide a single formal model that describes all data-flow analyses
  - Formalize the notions of "correctness," "conservativeness," and "optimality"
  - Correctness proof for DFAs
  - Place bounds on time complexity of iterative DFAs
  - This is not to drive the implementation,
    but to reason about data-flow analyses

Lattice

- Lattice L = (V, ≤):
  - V is a (possible infinite) set of elements
  - ≤ is a binary relation over elements of V
- Lower bound
  - z is a lower bound of x and y iff z ≤ x and z ≤ y
- Upper bound
  - z is a upper bound of x and y iff x ≤ z and y ≤ z
- Operations: meet (∧) and join (∨)
  - b ∨ c: least upper bound
  - b ∧ c: greater lower bound
  - An useful property: if e ≤ b and e ≤ c, then e ≤ b ∧ c



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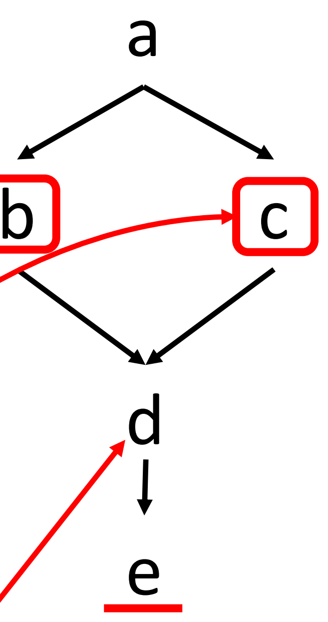
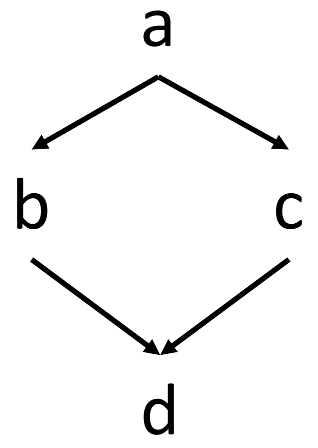


Lattice

- Lattice L = (V, ≤):
  - V is a (possible infinite) set of elements
  - ≤ is a binary relation over elements of V
- Properties of ≤:
  - ≤ is a partial order (reflexive, transitive, anti-symmetric)
  - Every pair of elements in V has
    - An unique **greatest lower bound** (a.k.a. meet) and
    - An unique **least upper bound** (a.k.a. join)
- Top (T) = unique greatest element of V (if it exists)
- Bottom (⊥) = unique least element of V (if it exists)
- Height of L: longest path from T to ⊥
  - Infinite large lattice can still have finite height



If you know nothing,
this is still a correct,
but conservative, solution

Lattice and DFA

- A lattice L = (V, ≤) describes all possible solutions of a given DFA
  - A lattice for reaching definitions
  - Another lattice for liveness analysis
  - …
  - For DFAs that look for solutions **per point** in the CFG, then one "lattice instance" per point

- The relation ≤ connects all solutions of its related DFA from the best one (T) to the worst one --most conservative one--(⊥)
  - Liveness analysis: *variables that might be used after a given point in the CFG*

  T = no variable is alive = { }

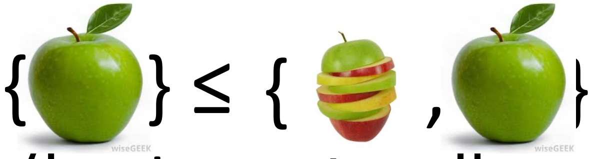
  Why? ⊥ = all variables are alive = V

- To solve a data-flow analysis: we traverse the lattice of a given DFA to find the correct solution **in a given point** of the CFG
  - We repeat it **for every point** in the CFG

Lattice example

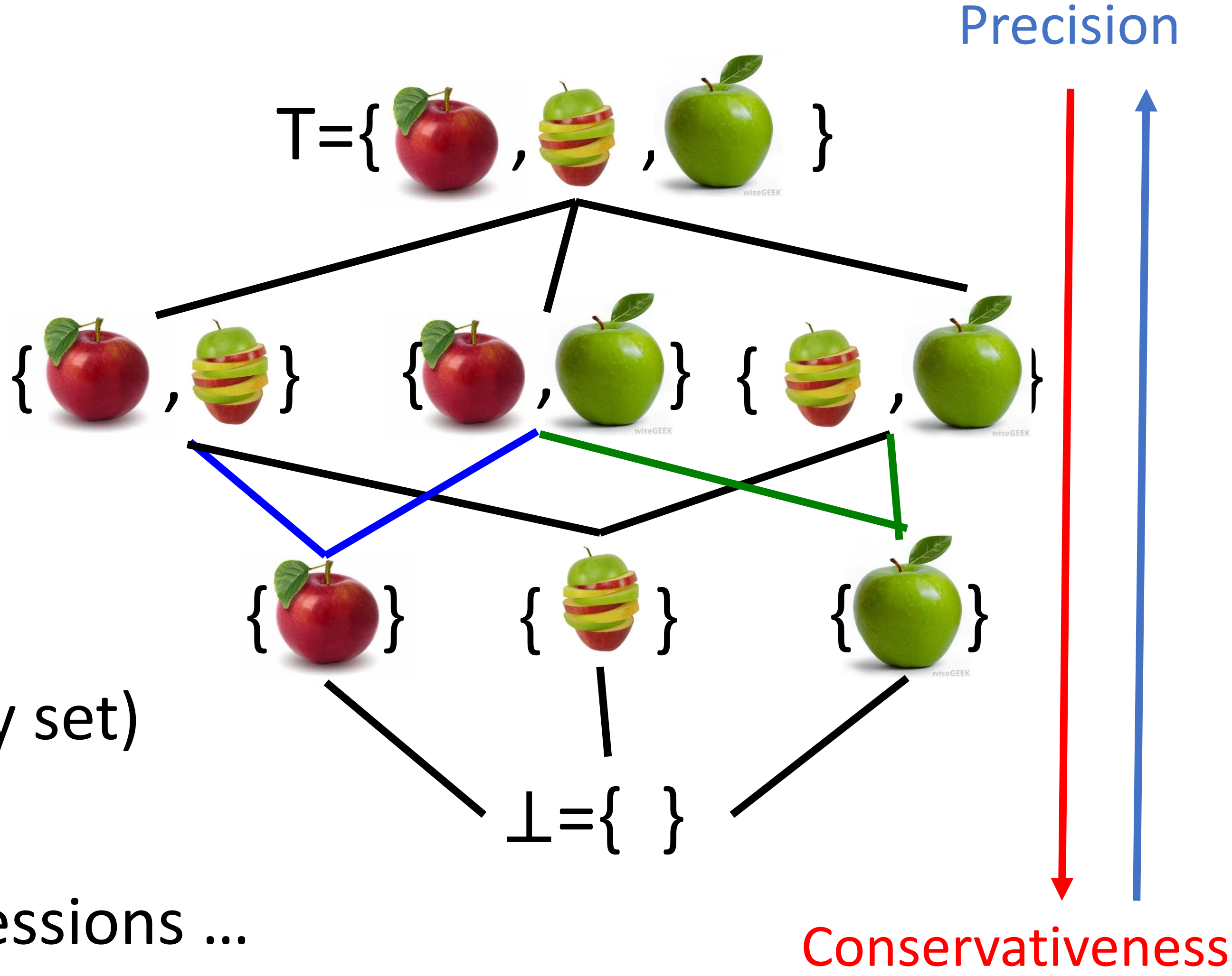
- How many apples I must have?
- V = sets of apples

- ≤ = set inclusion



- T = (best case) = all apples
- ⊥ = (worst case) no apples (empty set)

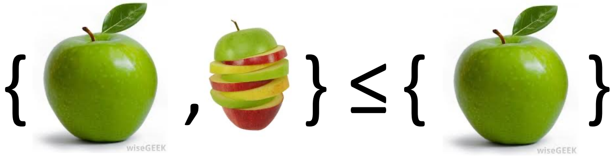
Apples, definitions, variables, expressions …



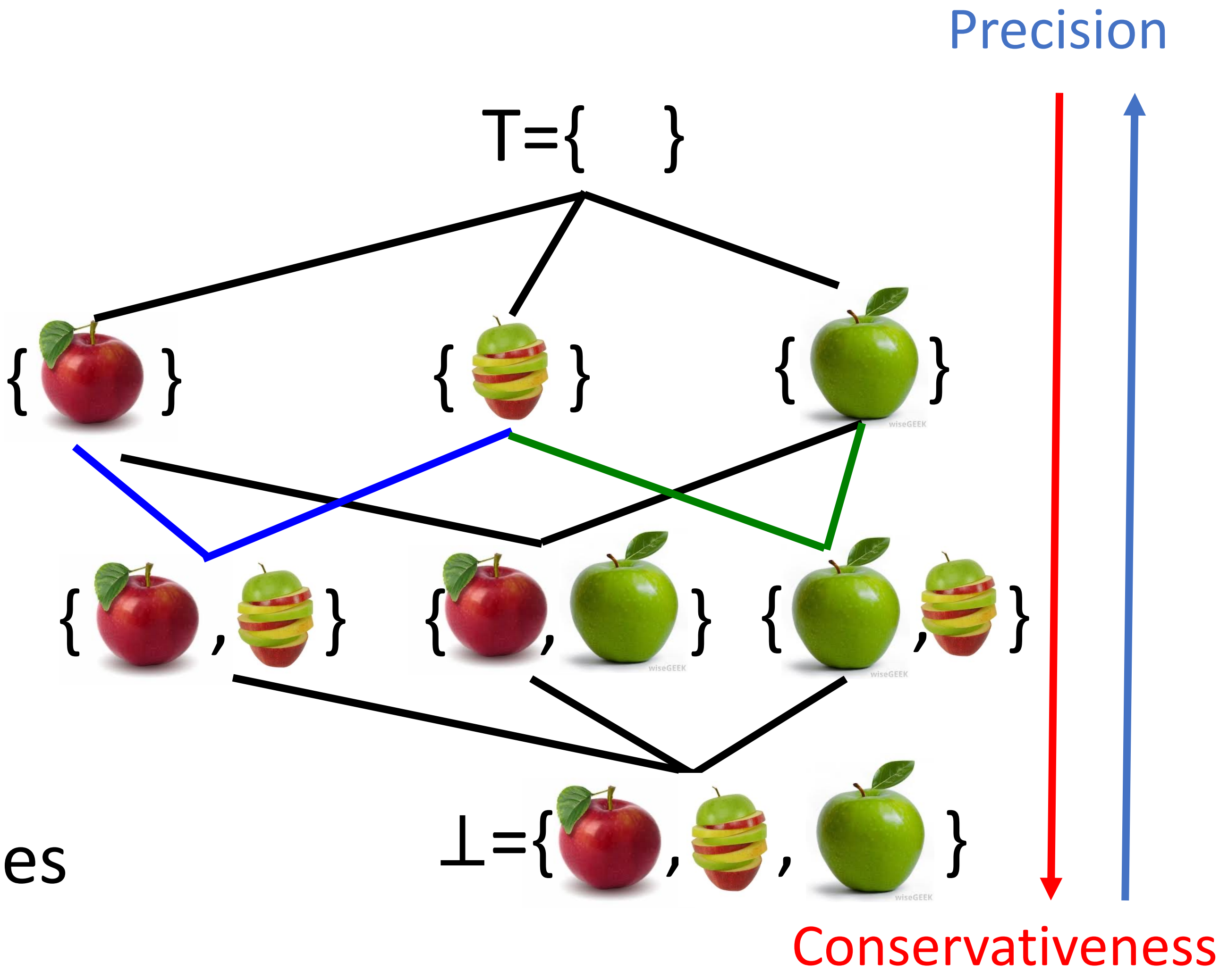
Another lattice example

- How many apples I may have?
- V = sets of apples

- ≤ = set inclusion



- T = no apples (empty set)
- ⊥ = (most conservative) all apples



How can we use this mathematical framework
, lattice,
to study a DFA?

Use of lattice for DFA

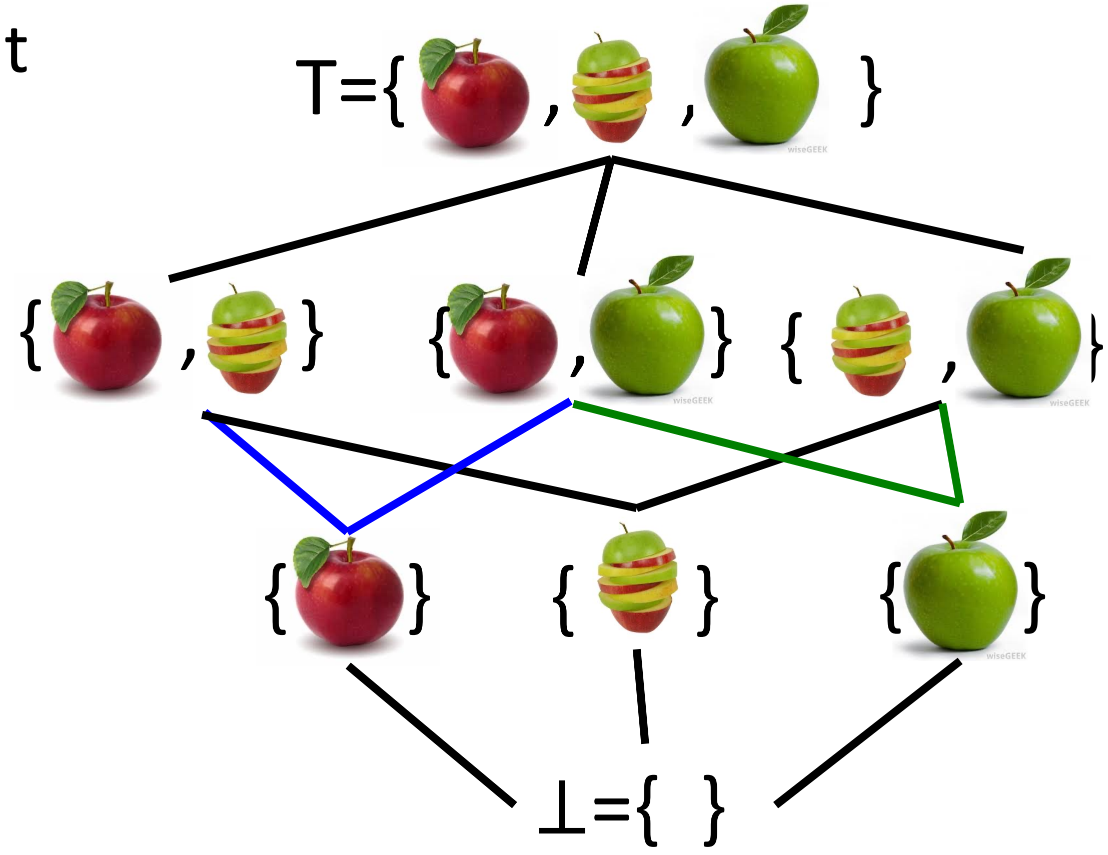
- Define domain of program properties (flow values --- apple sets) computed by data-flow analysis,
  and organize the domain of elements as a **lattice**

- Define how to traverse this domain to compute the final solution using lattice operations

- Exploit lattice theory in achieving goals

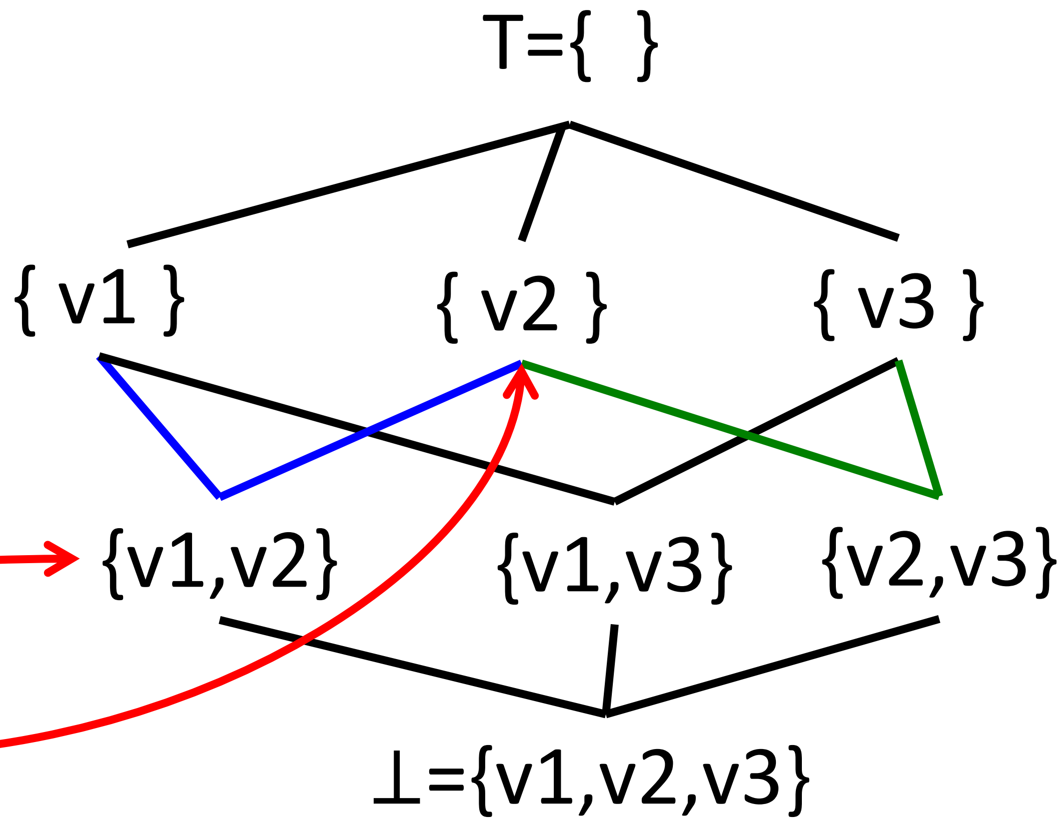
Data-flow analysis and lattice

- Elements of the lattice (V) represent flow values (e.g., an IN[] set)
  - *e.g.*, Sets of apples



Data-flow analysis and lattice

- Elements of the lattice (V) represent flow values (e.g., an IN[] set)
  - *e.g.*, Sets of live variables for liveness
- ⊥ “worst-case” information
  - *e.g.*, Universal set
- T “best-case” information
  - *e.g.*, Empty set
- If x ≤ y, then x is a conservative approximation of y
  - *e.g.*, Superset



Data-flow analysis and lattice (reaching defs)

- Elements of the lattice (V) represent flow values (IN[], OUT[])
  - *e.g.*, Sets of definitions
- T represents “best-case” information
  - *e.g.*, Empty set
- $\perp$ represents “worst-case” information
  - *e.g.*, Universal set
- If $x \leq y$, then x is a conservative approximation of y
  - *e.g.*, Superset

How do we choose

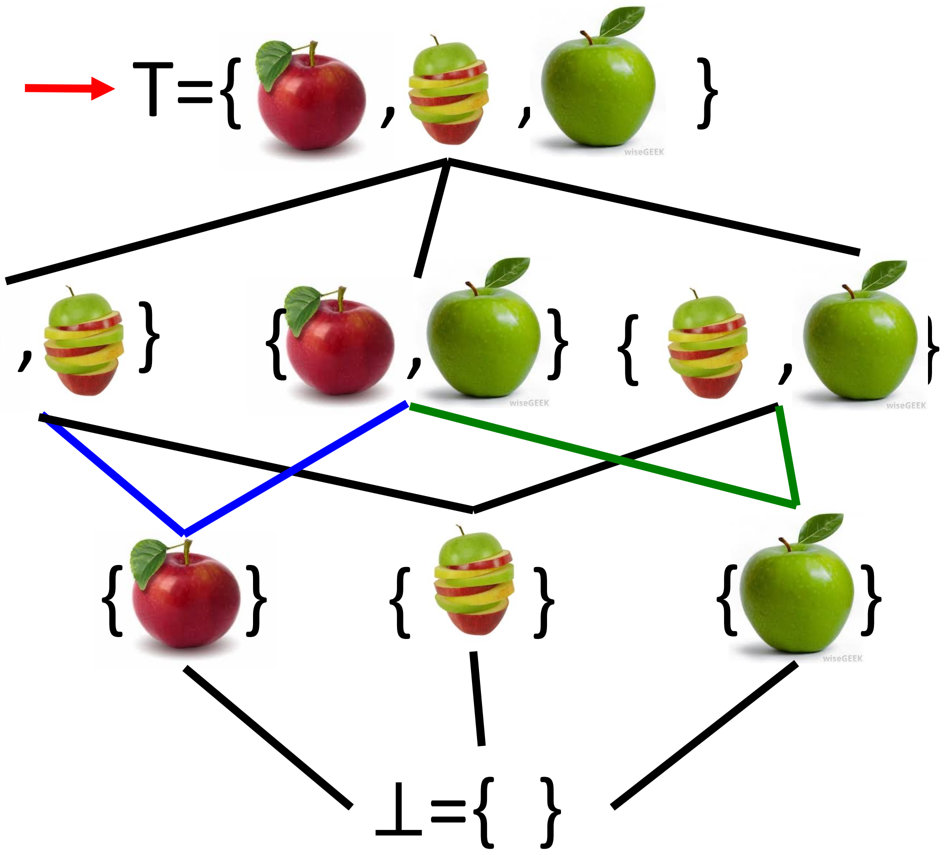
which element in our lattice

is the data-flow value

of a given point of the input program?

We traverse the lattice

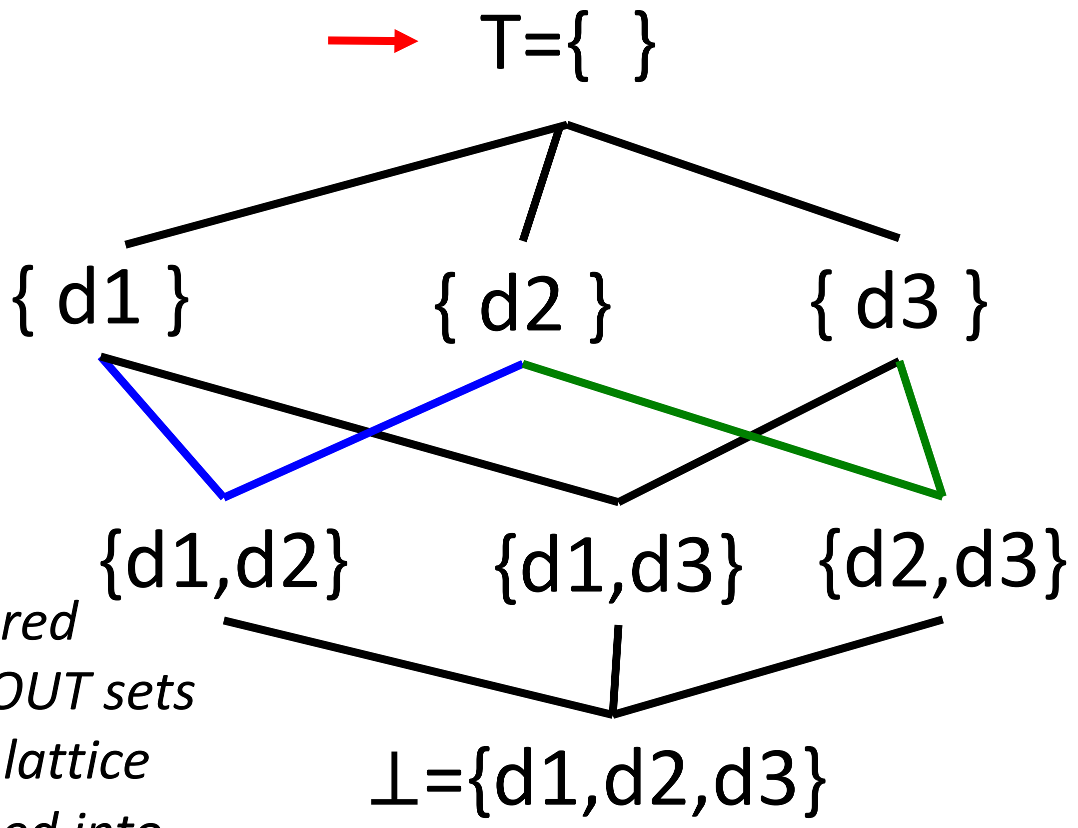
How many apples I must have?



We found out
there is no guarantee
we have the green apple

We traverse the lattice

for (each instruction *i* other than ENTRY) OUT[i] = { };



- *New information discovered while computing the IN/OUT sets will bring us down in the lattice*
- *New information is merged into the current knowledge/state/current-point-in-the-lattice*

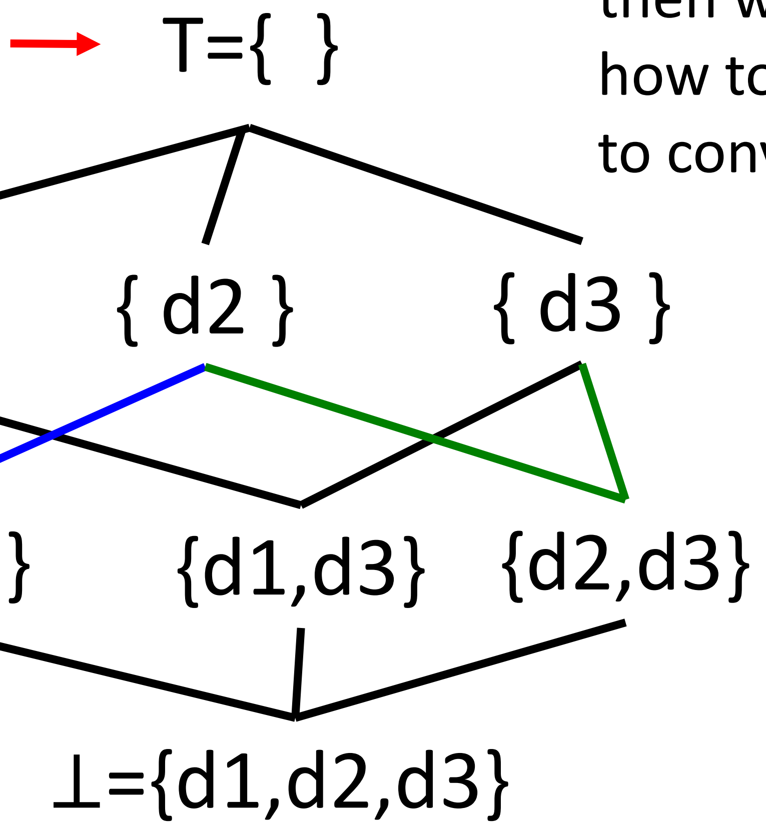
…let's see how

Merging information

- New information is found
  - e.g., a new definition (d1) reaches a given point in the CFG
- New information is described as a point in the lattice
  - e.g. {d1}
- We use the ”meet” operator ($\wedge$) of the lattice
  to merge the new information with the current one
  - e.g., set union
  - Current information: {d2}
  - New information: {d1}
  - Result: {d1} U {d2} = {d1, d2}

We traverse the lattice

As long as we know
how to get new information,
then we know
how to traverse the lattice
to converge to the final solution



We discover:
a new definition, d1,
reaches our point
in the CFG

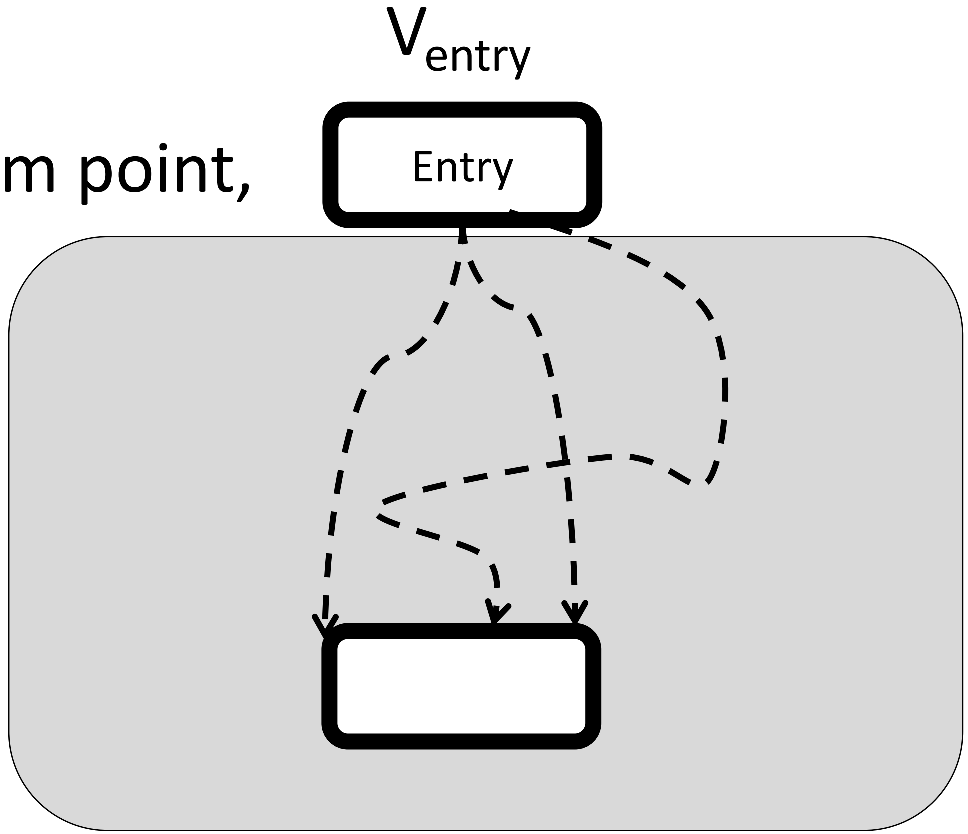
- New fact = {d1}
- $\{\,\} \wedge \{d1\} = \{d1\}$

How can we find new facts/information?

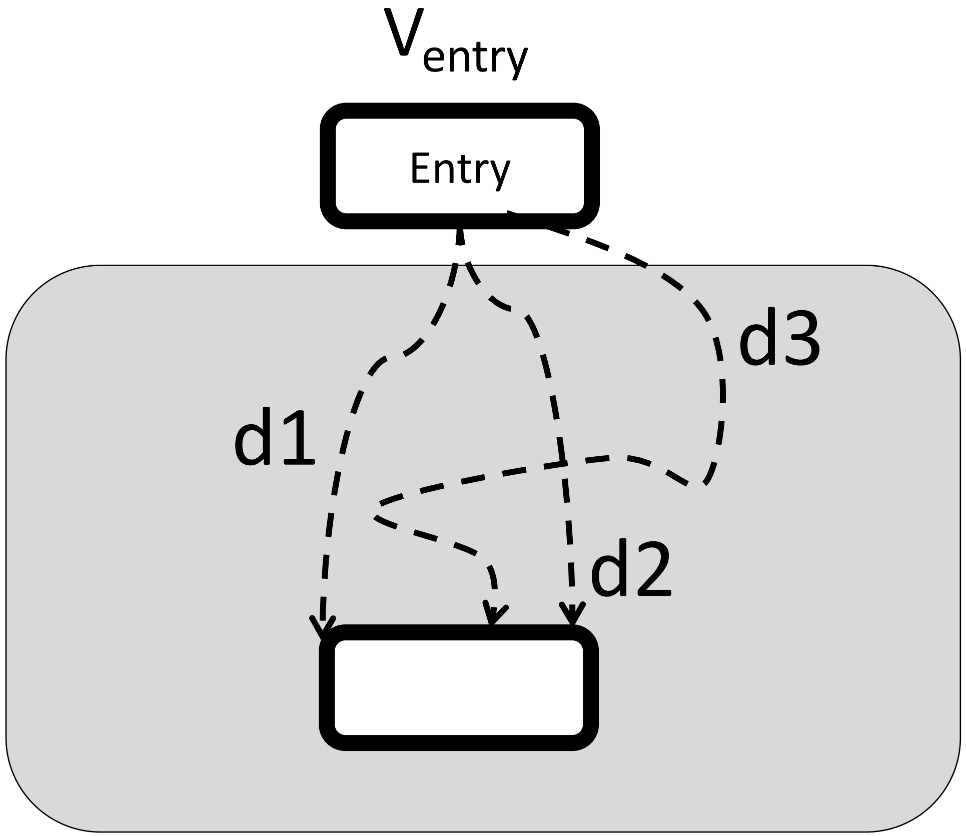
Computing a data-flow value (ideal)

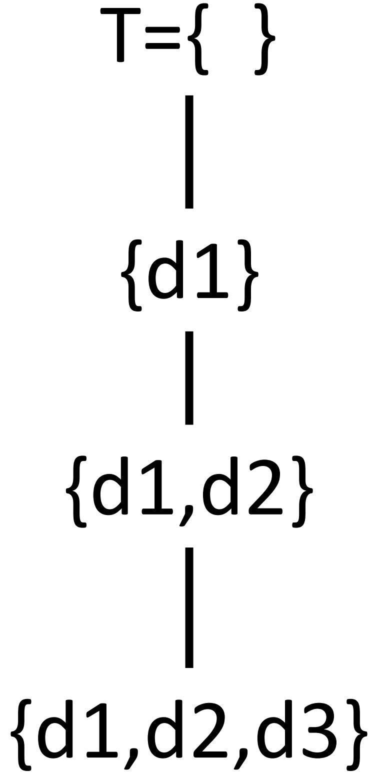
- For a forward problem,
  consider all possible paths
  from the entry to a given program point,
  compute the flow values
  at the end of each path,
  and then **meet** these values
  together

- Meet-over-all-paths (MOP)
  solution at each program point
  - It’s a correct solution



Computing MOP solution for reaching definitions

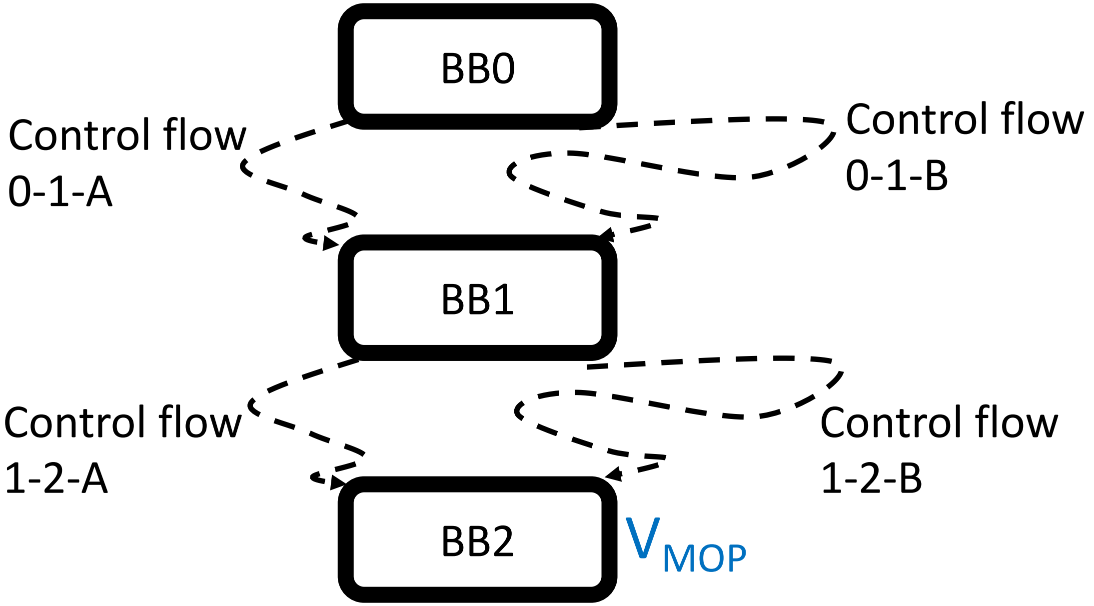




The problem of ideal solution

- **Problem**: all preceding paths must be analyzed
  - Exponential blow-up
- To compute the MOP solution in BB2:

0-1-A, 1-2-A
0-1-A, 1-2-B
0-1-B, 1-2-A
0-1-B, 1-2-B

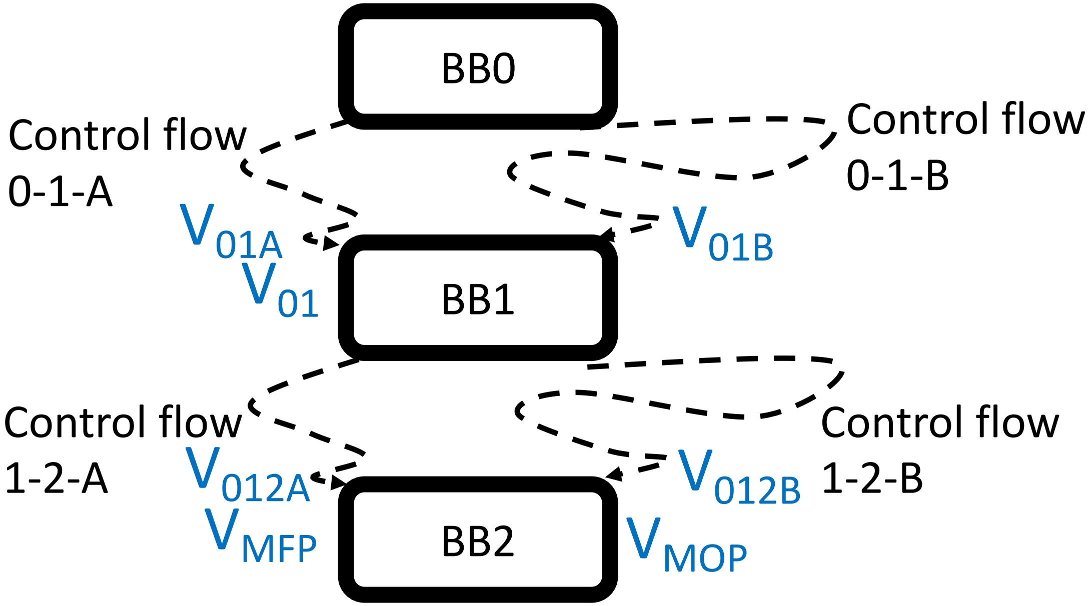


From ideal to practical solution

- **Problem**: all preceding paths must be analyzed
  - Exponential blow-up
- **Solution**: compute meets early (at merge points) rather than at the end
  - Maximum fixed-point (MFP)

$\boxed{\textbf{IN}[i] = \cup_{p \text{ a predecessor of } i} \textbf{OUT}[p];}$

- **Questions:**
  - Is MFP correct?
  - What's the precision of MFP?



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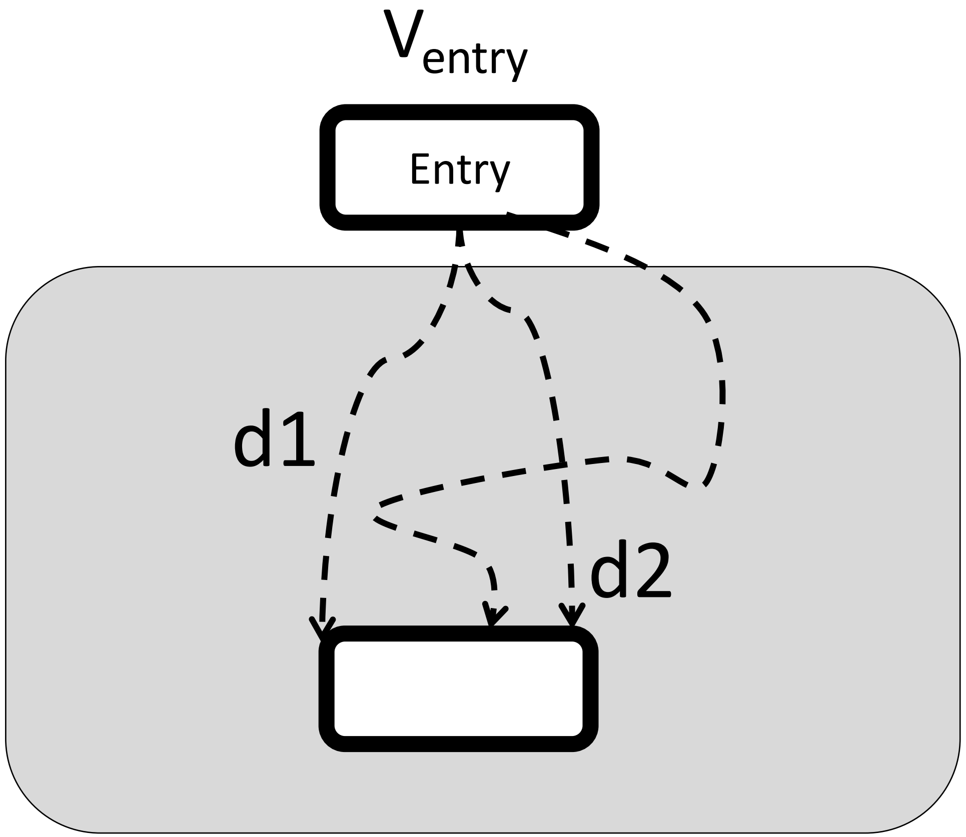
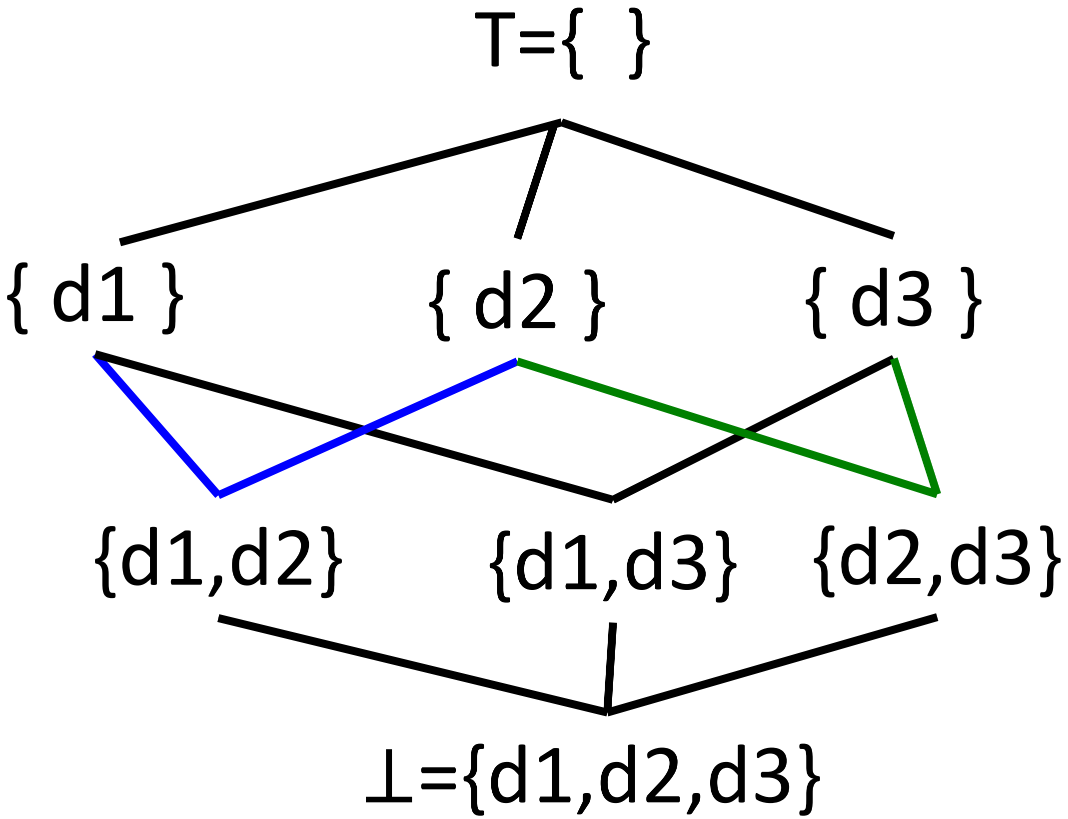
Correctness

$V_{correct} \leq V_{MOP}$



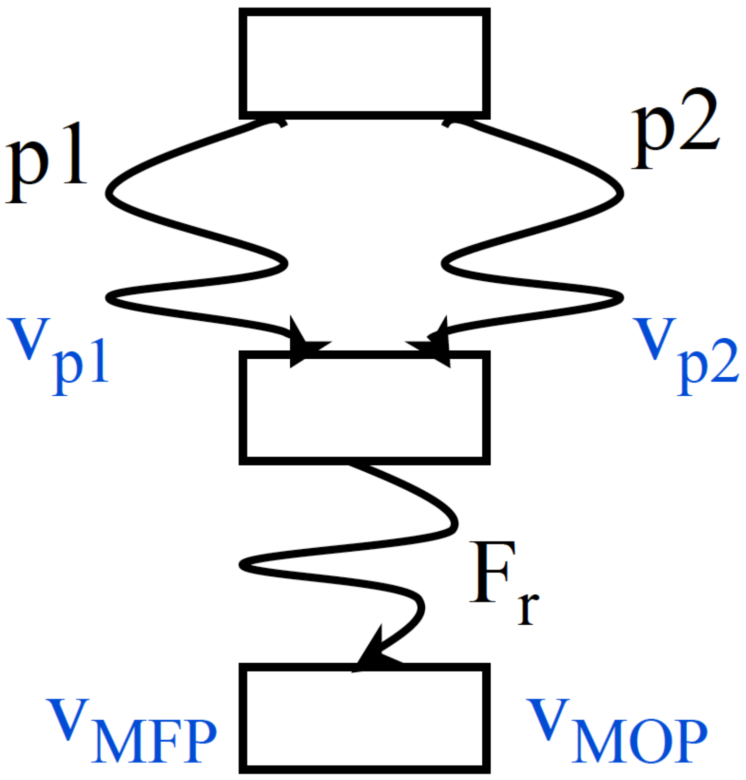


Correctness **fs is monotonic => MFP is correct!**

- Key idea:
  - "Is MFP correct?" iff $V_{MFP} \leq V_{MOP}$
- Focus on merges:
  - $V_{MOP} = fs(V_{p1}) \wedge fs(V_{p2})$
  - $V_{MFP} = fs(V_{p1} \wedge V_{p2})$

    Same function
  - $V_{MFP} \leq V_{MOP}$ iff $\underline{fs(V_{p1} \wedge V_{p2})} \leq fs(\underline{V_{p1}}) \wedge fs(\underline{V_{p2}})$

    Let us compare
- If fs is monotonic: $X \leq Y$ then $fs(X) \leq fs(Y)$
  - $(V_{p1} \wedge V_{p2}) \leq V_{p1}$ by definition of meet
  - $(V_{p1} \wedge V_{p2}) \leq V_{p2}$ by definition of meet
  - So $fs(V_{p1} \wedge V_{p2}) \leq fs(V_{p1})$ and $fs(V_{p1} \wedge V_{p2}) \leq fs(V_{p2})$
  - Therefore $fs(V_{p1} \wedge V_{p2}) \leq fs(V_{p1}) \wedge fs(V_{p2})$
  - And therefore $V_{MFP} \leq V_{MOP}$



An useful property: if $e \leq b$ and $e \leq c$, then $e \leq b \wedge c$

Monotonicity

- $X \leq Y$ then $fs(X) \leq fs(Y)$

- If the flow function f is applied to two members of V,
  the result of applying f to the "lesser" of the two members
  will be under the result of applying f to the "greater" of the two

- More conservative inputs
  leads to more conservative outputs
  (never more optimistic outputs)

Convergence

- **From lattice theory**
  If *fs* is monotonic,
  then the maximum number of times *fs* can be applied
  w/o reaching a fixed point is Height(V) – 1

- Iterative DFA is guaranteed to terminate
  if the fs is monotonic and
  the lattice has finite height

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Precision

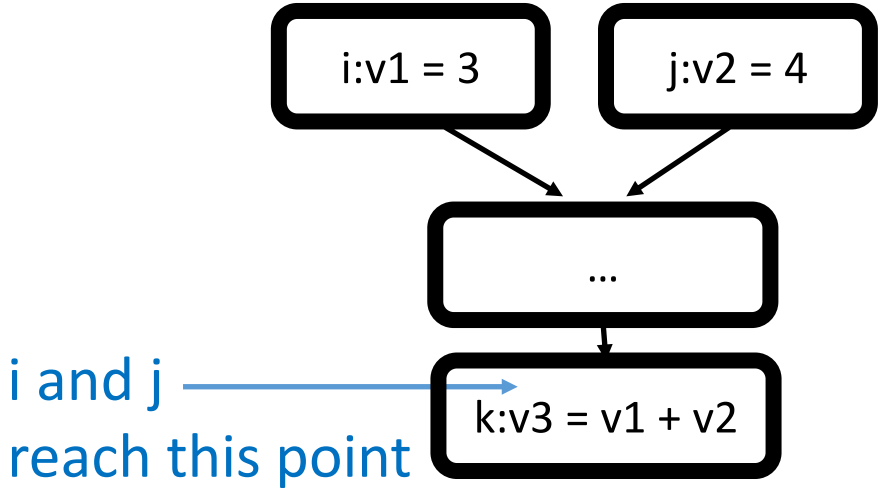
- V_{MOP}: the best solution
- $V_{MFP} \leq V_{MOP}$
  - $fs(V_{p1} \wedge V_{p2}) \leq fs(V_{p1}) \wedge fs(V_{p2})$

* is distributive over +

4 * (2 + 3) = 4 * (5) = 20

(4 * 2) + (4 * 3) = 8 + 12 = 20

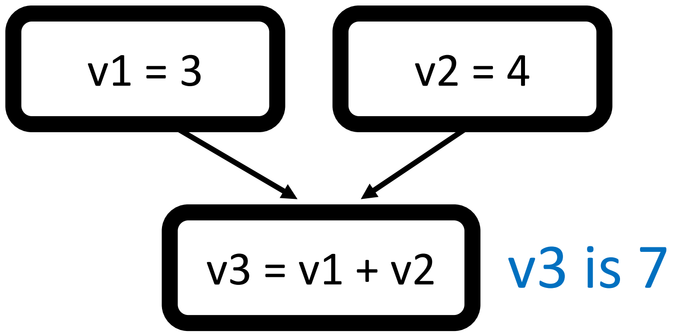
- Distributive fs over $\wedge$
  - $fs(V_{p1} \wedge V_{p2}) = fs(V_{p1}) \wedge fs(V_{p2})$
  - $V_{MFP} = V_{MOP}$



- Is reaching definition fs distributive?
  - (did having performed $\wedge$ earlier change anything?)

A new DFA example: reaching constants

- Goal
  - Compute the value that a variable must have at a program point (no SSA)
- Flow values (V)
  - Set of (variable,constant) pairs
- Merge function
  - Intersection
- Data-flow equations
  - Effect of node n: x = c
    - KILL[n] = {(x,k)| ∀k}
    - GEN[n] = {(x,c)}
  - Effect of node n: x = y + z
    - KILL[n] = {(x,k)| ∀k}
    - GEN[n] = {(x,c) | c=valy+valz, (y, valy) ∈ IN[n], (z, valz) ∈ IN[n]}



Reaching constants: characteristics

- $\perp$ = ?
- IN = ?
- OUT = ?
- Let's study this analysis
  - Does it convergence?
    - is *fs* monotonic? Has the lattice a finite height?
  - What is the precision of the solution?
    - is fs distributive?

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Complexity

```
OUT[ENTRY] = { };
for (each instruction i other than ENTRY)  OUT[i] = { };
do {
  for (each instruction i other than ENTRY) {
    IN[i] = ∪_{p a predecessor of i} OUT[p];
    OUT[i] = GEN[i] ∪ (IN[i] – KILL[i]);
  }
} while (changes to any OUT occur);
```

Complexity

- N instructions (N definitions at most)
  - Each IN/OUT set has at most N elements
  - Each set-union operation takes O(N) time
  - The complexity of a single invocation of for
    - constant # of set operations per node
    - O(N) nodes $\Rightarrow$ $O(N^2)$ time for the loop
  - Each invocation of the for loop can only make the IN/OUT sets larger
  - Each invocation modifies in the worst case only one set $\Rightarrow$ $O(N^3)$
  - N invocations to reach the fixed point at most
- Worst case: $O(N^4)$
- Typical case: 2 to 3 invocations with good ordering & sparse sets
  - Between N and N^2

N=500
Worst case: 62,500,000,000
Optimized average case:
500 – 250,000

Optimization: basic blocks

```
OUT[ENTRY] = { };
for (each basic block B other than ENTRY)  OUT[B] = { };
do {
  for (each basic block B other than ENTRY) {
    IN[B] = ∪_{p a predecessor of B} OUT[p];
    OUT[B]= GEN[B]∪ (IN[B] – KILL[B]);
  }
} while (changes to any OUT occur);
```

Optimization: work list

OUT[ENTRY] = { };

for (each basic block B other than ENTRY) OUT[B] = { };

workList = all basic blocks

while (workList isn't empty)

  B = pick and remove a block from workList

  oldOUT = OUT[B]

  IN[B] = $\cup_{p \text{ a predecessor of } B}$ OUT[p];

  OUT[B]= GEN[B]$\cup$ (IN[B] – KILL[B]);

  if (oldOut != OUT[B]) workList = workList U {all successors of B}

}