

Typing Rules

What are type systems made of?

- Type systems are formal systems
- Usually described using logical rules
 - Provide a *specification* for what is well-formed
 - Rather than an *implementation* for determining whether something is well-formed
- If we want a type *checker* we turn these rules into code
 - Can be easy or hard, depending on the type system

Type Relation

We are defining a *type judgment*

$$\Gamma \vdash \mathbf{e} : \tau$$

Which means: with the type bindings in Γ , I can conclude that \mathbf{e} has the type τ

Which implies that \mathbf{e} is well-formed

Γ is the *type environment*: a map from $\langle \mathbf{id} \rangle$ to τ (type)

E.g., [$\mathbf{x} \leftarrow \mathit{boolean}, \mathbf{y} \leftarrow \mathit{number}$]

Similar in spirit to a deferred substitution, but maps to types, not values

Inference rules

Judgments are typically defined as a set of inference rules

$$\frac{A \quad B}{C}$$

This is a *rule*, which says: If I know A and B, then I can conclude C

- There could be 0 or more things above the bar

Terminology: A and B are *premises* C is the *conclusion*

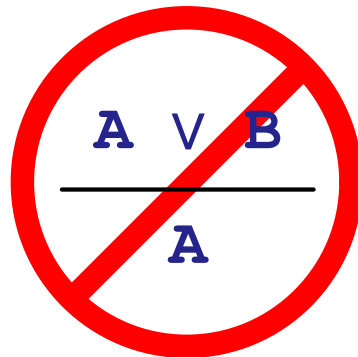
Inference Rule Example: Logical Or

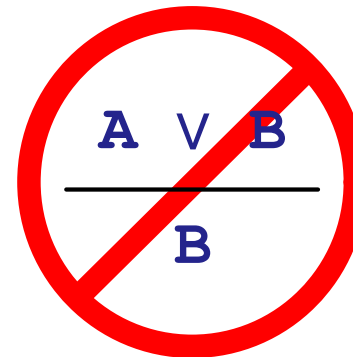
These two rules *together* describe logical or

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

These two are false inferences


$$\frac{A \vee B}{A}$$


$$\frac{A \vee B}{B}$$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \text{number}$

$\Gamma \vdash \text{true} : \text{boolean}$

$\Gamma \vdash \text{false} : \text{boolean}$

$\Gamma \vdash e_1 : \text{number} \quad \Gamma \vdash e_2 : \text{number}$

$\Gamma \vdash \{+ e_1 e_2\} : \text{number}$

(These have no premises)

$1 : \text{number}$

$\text{true} : \text{boolean}$

$1 : \text{number} \quad 2 : \text{number}$

$\{+ 1 2\} : \text{number}$

$1 : \text{number} \quad \text{false} : \text{boolean}$

$\{+ 1 \text{false}\} : \text{no type}$

Type Rules

$\Gamma \vdash \langle \text{num} \rangle : \textit{number}$

$\Gamma \vdash \text{true} : \textit{boolean}$

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$\Gamma \vdash \{+ e_1 e_2\} : \textit{number}$

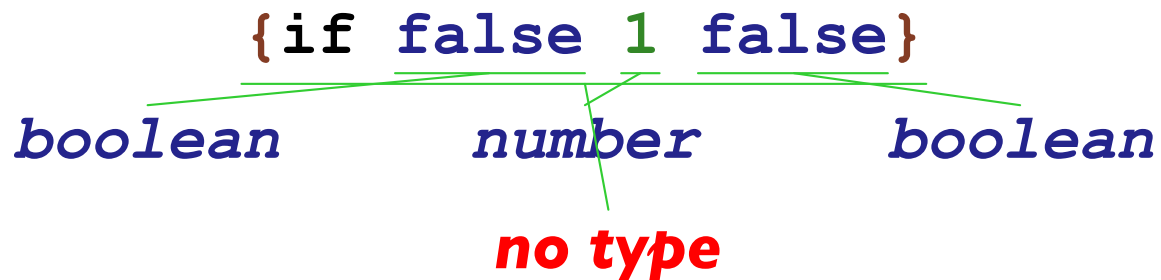
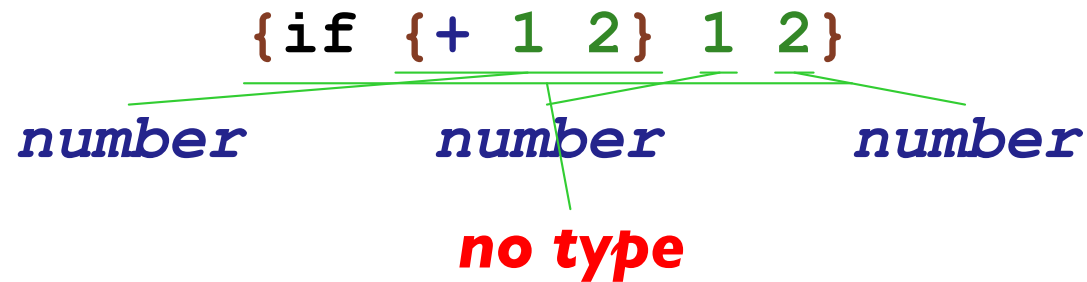
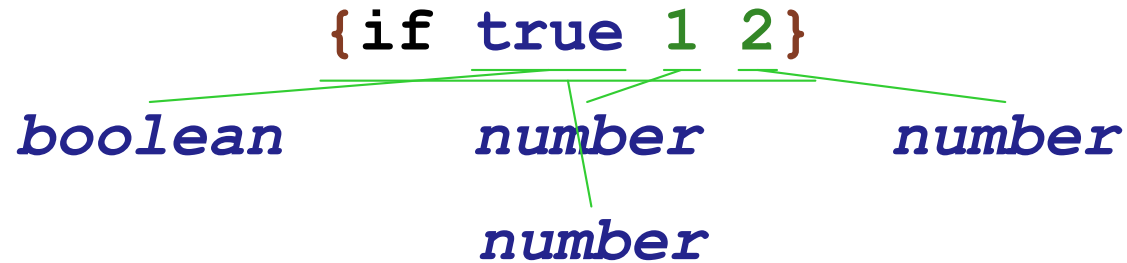
(These have no premises)

$1 : \textit{number} \quad 2 : \textit{number}$

$\{+ 1 2\} : \textit{number} \quad 3 : \textit{number}$

$\{+ \{+ 1 2\} 3\} : \textit{number}$

Types: Conditionals



Conditional Type Rules

$$\Gamma \vdash e_1 : \textit{boolean} \quad \Gamma \vdash e_2 : \tau_0 \quad \Gamma \vdash e_3 : \tau_0$$

$$\Gamma \vdash \{\textit{if } e_1 \ e_2 \ e_3\} : \tau_0$$
$$\textit{true} : \textit{boolean} \quad 1 : \textit{number} \quad 2 : \textit{number}$$

$$\{\textit{if } \textit{true} \ 1 \ 2\} : \textit{number}$$
$$\{+ \ 1 \ 2\} : \textit{number} \quad 1 : \textit{number} \quad 2 : \textit{number}$$

$$\{\textit{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \textit{no type}$$
$$\textit{false} : \textit{boolean} \quad 1 : \textit{number} \quad \textit{false} : \textit{boolean}$$

$$\{\textit{if } \textit{false} \ 1 \ \textit{false}\} : \textit{no type}$$

Types: Variables and Functions

x : no type

`{fun {x : boolean} x}`

boolean

(boolean → boolean)

`{fun {x : boolean} {if x 1 2}}`

boolean

number

number

number

(boolean → number)

Variable and Function Type Rules

$$[\dots \langle \text{id} \rangle \leftarrow \tau \dots] \vdash \langle \text{id} \rangle : \tau$$
$$\Gamma [\langle \text{id} \rangle \leftarrow \tau_1] \vdash \mathbf{e} : \tau_2$$

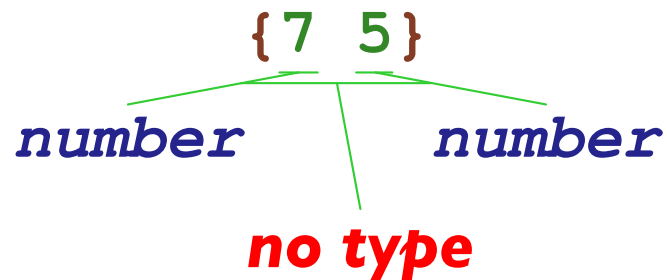
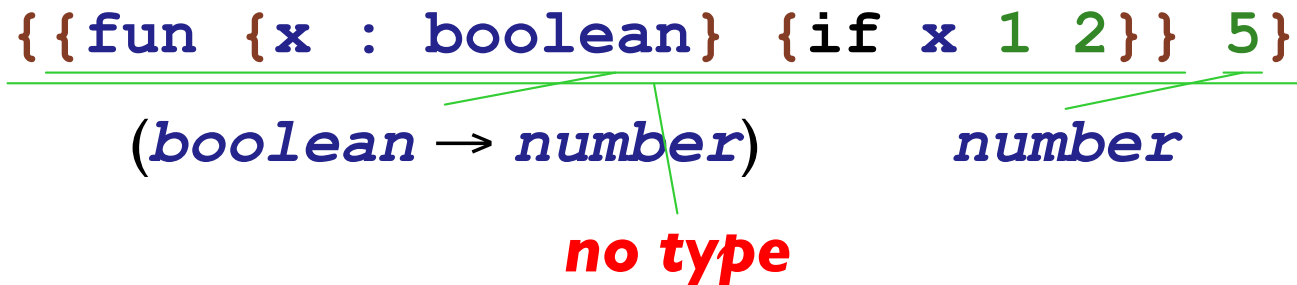
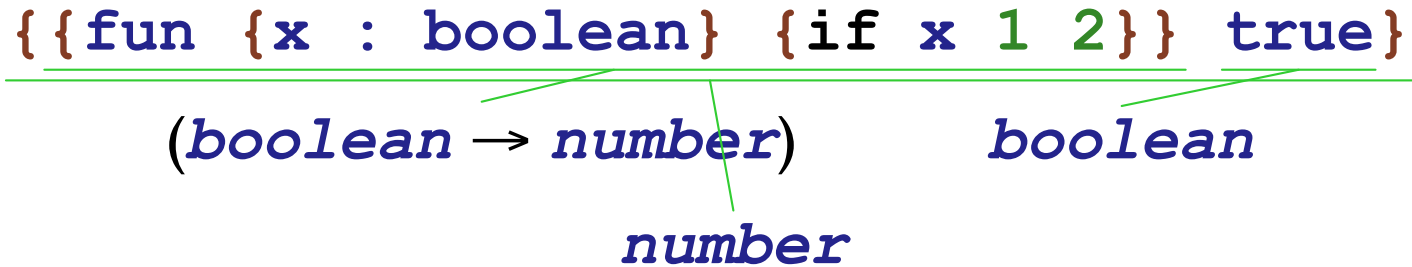
$$\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle : \tau_1 \} \mathbf{e} \} : (\tau_1 \rightarrow \tau_2)$$
$$\emptyset \vdash \mathbf{x} : \text{no type}$$
$$[\mathbf{x} \leftarrow \text{boolean}] \vdash \mathbf{x} : \text{boolean}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{boolean} \} \mathbf{x} \} : (\text{boolean} \rightarrow \text{boolean})$$
$$[\mathbf{x} \leftarrow \text{boolean}] \vdash \mathbf{x} : \text{boolean} \quad [\mathbf{x} \leftarrow \text{boolean}] \vdash \mathbf{1} : \text{number} \quad [\mathbf{x} \leftarrow \text{boolean}] \vdash \mathbf{2} : \text{number}$$

$$[\mathbf{x} \leftarrow \text{boolean}] \vdash \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} : \text{number}$$

$$\emptyset \vdash \{ \text{fun } \{ \mathbf{x} : \text{boolean} \} \{ \text{if } \mathbf{x} \ \mathbf{1} \ \mathbf{2} \} \} : (\text{boolean} \rightarrow \text{number})$$

Types: Function Calls



Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash \text{true} : \text{boolean}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{number}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} : (\text{boolean} \rightarrow \text{number}) \quad \emptyset \vdash 5 : \text{number}}{\emptyset \vdash \{\{\text{fun } \{x : \text{boolean}\} \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{number} \quad \emptyset \vdash 5 : \text{number}}{\emptyset \vdash \{7 \ 5\} : \text{no type}}$$

Types: Multiple Arguments

`{fun {x : number y : number} {+ x y}}`

number *number*
number

(number number → number)

`{{fun {x : number y : number} {+ x y}} 5 6}`

(number number → number) *number* *number*
number

`{{fun {x : number y : number} {+ x y}} 5}`

(number number → number) *number*

no type

Revised Function and Call Rules

$$\frac{\Gamma[\langle \text{id} \rangle_1 \leftarrow \tau_1 \dots \langle \text{id} \rangle_n \leftarrow \tau_n] \vdash \mathbf{e} : \tau_0}{\Gamma \vdash \{ \text{fun } \{ \langle \text{id} \rangle_1 : \tau_1 \dots \langle \text{id} \rangle_n : \tau_n \} \mathbf{e} \} : (\tau_1 \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \mathbf{e}_0 : (\tau_1 \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \mathbf{e}_1 : \tau_1 \quad \dots \quad \Gamma \vdash \mathbf{e}_n : \tau_n}{\Gamma \vdash \{ \mathbf{e}_0 \ \mathbf{e}_1 \ \dots \ \mathbf{e}_n \} : \tau_0}$$