# Typing Rules

# What are type systems made of?

- Type systems are formal systems
- Usually described using logical rules
  - Provide a *specification* for what is well-formed
  - Rather than an *implementation* for determining whether something is well-formed
- If we want a type *checker* we turn these rules into code
  - Can be easy or hard, depending on the type system

# Type Relation

We are defining a *type judgment*

$$\Gamma \vdash \mathbf{e} : \tau$$

Which means: with the type bindings in $\Gamma$, I can conclude that **e** has the type $\tau$

Which implies that **e** is well-formed

$\Gamma$ is the *type environment*: a map from `<id>` to $\tau$ (type)

E.g., [ **x**←***boolean***, **y**←***number*** ]

Similar in spirit to a deferred substitution, but maps to types, not values

# Inference rules

Judgments are typically defined as a set of inference rules

$$\frac{\mathbf{A} \qquad \mathbf{B}}{\mathbf{C}}$$

This is a *rule*, which says: If I know A and B, then I can conclude C

- There could be 0 or more things above the bar

Terminology: A and B are *premises* C is the *conclusion*

# Inference Rule Example: Logical Or

These two rules *together* describe logical or

$$\frac{\mathbf{A}}{\mathbf{A} \lor \mathbf{B}} \qquad\qquad \frac{\mathbf{B}}{\mathbf{A} \lor \mathbf{B}}$$

These two are false inferences

$$\frac{\mathbf{A} \lor \mathbf{B}}{\mathbf{A}} \qquad\qquad \frac{\mathbf{A} \lor \mathbf{B}}{\mathbf{B}}$$

# Type Rules

$\Gamma \vdash$ `<num>` : *number*

$\Gamma \vdash$ `true` : *boolean*

$\Gamma \vdash$ `false` : *boolean*

(These have no premises)

$$\frac{\Gamma \vdash \mathbf{e}_1 : \mathit{number} \qquad \Gamma \vdash \mathbf{e}_2 : \mathit{number}}{\Gamma \vdash \texttt{\{+ } \mathbf{e}_1\ \mathbf{e}_2\texttt{\}} : \mathit{number}}$$

`1` : *number*

`true` : *boolean*

$$\frac{\texttt{1} : \mathit{number} \qquad \texttt{2} : \mathit{number}}{\texttt{\{+ 1 2\}} : \mathit{number}}$$

$$\frac{\texttt{1} : \mathit{number} \qquad \texttt{false} : \mathit{boolean}}{\texttt{\{+ 1 false\}} : \textit{no type}}$$

# Type Rules

$\Gamma \vdash$ `<num>` $:$ *number*

$\Gamma \vdash$ `true` $:$ *boolean*

$\Gamma \vdash$ `false` $:$ *boolean*

(These have no premises)

$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{number} \qquad \Gamma \vdash \mathbf{e}_2 : \textit{number}}{\Gamma \vdash \texttt{\{+ } \mathbf{e}_1\ \mathbf{e}_2\texttt{\}} : \textit{number}}$$

$$\frac{\dfrac{\texttt{1} : \textit{number} \qquad \texttt{2} : \textit{number}}{\texttt{\{+ 1 2\}} : \textit{number}} \qquad \texttt{3} : \textit{number}}{\texttt{\{+ \{+ 1 2\} 3\}} : \textit{number}}$$

# Types: Conditionals

`{if true 1 2}`

*boolean* *number* *number*

*number*

`{if {+ 1 2} 1 2}`

*number* *number* *number*

***no type***

`{if false 1 false}`

*boolean* *number* *boolean*

***no type***

# Conditional Type Rules

$$\frac{\Gamma \vdash \mathbf{e}_1 : \mathit{boolean} \qquad \Gamma \vdash \mathbf{e}_2 : \tau_0 \qquad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \texttt{\{if } \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\texttt{\}} : \tau_0}$$

$$\frac{\texttt{true} : \mathit{boolean} \qquad \texttt{1} : \mathit{number} \qquad \texttt{2} : \mathit{number}}{\texttt{\{if true 1 2\}} : \mathit{number}}$$

$$\frac{\texttt{\{+ 1 2\}} : \mathit{number} \qquad \texttt{1} : \mathit{number} \qquad \texttt{2} : \mathit{number}}{\texttt{\{if \{+ 1 2\} 1 2\}} : \textit{no type}}$$

$$\frac{\texttt{false} : \mathit{boolean} \qquad \texttt{1} : \mathit{number} \qquad \texttt{false} : \mathit{boolean}}{\texttt{\{if false 1 false\}} : \textit{no type}}$$

# Types: Variables and Functions

**x : *no type***

**{fun {x : boolean} x}**

***boolean***

(***boolean*** → ***boolean***)

**{fun {x : boolean} {if x 1 2}}**

***boolean*** ***number*** ***number***

***number***

(***boolean*** → ***number***)

# Variable and Function Type Rules

$$[\ \ldots\ \texttt{<id>} \leftarrow \tau\ \ldots\ ] \vdash \texttt{<id>} : \tau$$

$$\frac{\Gamma[\ \texttt{<id>} \leftarrow \tau_1\ ] \vdash \texttt{e} : \tau_2}{\Gamma \vdash \texttt{\{fun \{<id> : } \tau_1 \texttt{\} e\}} : (\tau_1 \rightarrow \tau_2)}$$

$$\emptyset \vdash \texttt{x} : \textit{no type}$$

$$\frac{[\ \texttt{x} \leftarrow \textit{boolean}\ ] \vdash \texttt{x} : \textit{boolean}}{\emptyset \vdash \texttt{\{fun \{x : boolean\} x\}} : (\textit{boolean} \rightarrow \textit{boolean})}$$

$$\frac{\dfrac{[\ \texttt{x} \leftarrow \textit{boolean}\ ] \vdash \texttt{x} : \textit{boolean} \quad [\ \texttt{x} \leftarrow \textit{boolean}\ ] \vdash \texttt{1} : \textit{number} \quad [\ \texttt{x} \leftarrow \textit{boolean}\ ] \vdash \texttt{2} : \textit{number}}{[\ \texttt{x} \leftarrow \textit{boolean}\ ] \vdash \texttt{\{if x 1 2\}} : \textit{number}}}{\emptyset \vdash \texttt{\{fun \{x : boolean\} \{if x 1 2\}\}} : (\textit{boolean} \rightarrow \textit{number})}$$

# Types: Function Calls

`{{fun {x : boolean} {if x 1 2}} true}`

*(boolean → number)*   *boolean*

*number*

`{{fun {x : boolean} {if x 1 2}} 5}`

*(boolean → number)*   *number*

***no type***

`{7 5}`

*number*   *number*

***no type***

# Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \qquad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1\ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \texttt{\{fun \{x : boolean\} \{if x 1 2\}\}} : (\textit{boolean} \rightarrow \textit{number}) \qquad \emptyset \vdash \texttt{true} : \textit{boolean}}{\emptyset \vdash \texttt{\{\{fun \{x : boolean\} \{if x 1 2\}\} true\}} : \textit{number}}$$

$$\frac{\emptyset \vdash \texttt{\{fun \{x : boolean\} \{if x 1 2\}\}} : (\textit{boolean} \rightarrow \textit{number}) \qquad \emptyset \vdash \texttt{5} : \textit{number}}{\emptyset \vdash \texttt{\{\{fun \{x : boolean\} \{if x 1 2\}\} 5\}} : \textbf{\textit{no type}}}$$

$$\frac{\emptyset \vdash 7 : \textit{number} \qquad \emptyset \vdash 5 : \textit{number}}{\emptyset \vdash \{7\ \ 5\} : \textbf{\textit{no type}}}$$

# Types: Multiple Arguments

`{fun {x : number y : number} {+ x y}}`

*number* *number*

*number*

(*number number* → *number*)

`{{fun {x : number y : number} {+ x y}} 5 6}`

(*number number* → *number*) *number* *number*

*number*

`{{fun {x : number y : number} {+ x y}} 5}`

(*number number* → *number*) *number*

**no type**

# Revised Function and Call Rules

$$\frac{\Gamma[\texttt{<id>}_1 \leftarrow \tau_1 \ldots \texttt{<id>}_n \leftarrow \tau_n] \vdash \texttt{e} : \tau_0}{\Gamma \vdash \texttt{\{fun \{<id>}_1 : \tau_1 \ \ldots \ \texttt{<id>}_n : \tau_n\texttt{\} e\}} : (\tau_1 \ldots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \texttt{e}_0 : (\tau_1 \ldots \tau_n \rightarrow \tau_0) \qquad \Gamma \vdash \texttt{e}_1 : \tau_1 \qquad \ldots \qquad \Gamma \vdash \texttt{e}_n : \tau_n}{\Gamma \vdash \texttt{\{e}_0 \ \texttt{e}_1 \ \ldots \ \texttt{e}_n\texttt{\}} : \tau_0}$$